第八屆培正數學邀請賽

8th Pui Ching Invitational Mathematics Competition

決賽（中一組）

Final Event (Secondary 1)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

1. 本卷共設 20 題，總分為 100 分。

   There are 20 questions in this paper and the total score is 100.

2. 除特別指明外，本卷內的所有數均為十進制。

   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

   Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。

   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。

   The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 網站「數學資料庫」於 2003 年 3 月 14 日成立。2009 年 3 月 14 日（星期六）是「數學資料庫」成立六週年的紀念日。「數學資料庫」下一次在星期六的週年紀念日出現在哪一年？

   The website ‘Mathematical Database’ was founded on 14th March 2003. On Saturday 14th March 2009, ‘Mathematical Database’ celebrates its 6th anniversary. In which year will ‘Mathematical Database’ next celebrate its anniversary on a Saturday?

2. 下表詳列了某城市的用水收費。該市的一所理髮店毗鄰一所完全不用水的書店，書店東主決定以每立方米 8 元的價格把水賣給理髮店。假設除水費外，書店並沒有其他成本，那麼書店賣水每月最多可獲多少元的利潤？

   The following table lists the water charges of a city. A barber shop in the city is next to a bookstore which does not consume any water. The owner of the bookstore decides to sell water to the barber shop at a price 8 dollars per cubic metre. Assume that there is no cost for the bookstore except the water charges, what is the maximum possible monthly profit (in dollars) for the bookstore by selling water?

每月用水收費表	
首 10 立方米	免費
次 20 立方米	每立方米 2 元
次 30 立方米	每立方米 5 元
次 40 立方米	每立方米 9 元
	以後每立方米 15 元

Table for monthly fee of water supply	
The first 10 cubic metres	Free
The next 20 cubic metres	2 dollars per cubic metre
The next 30 cubic metres	5 dollars per cubic metre
The next 40 cubic metres	9 dollars per cubic metre
	15 dollars per cubic metre thereafter

3. 現有一個正六邊形，並容許從中選三個頂點來組成三角形。那麼共可組成多少個不同的三角形？（兩個全等的三角形視爲同一個三角形。）

   If one is allowed to take three vertices from a given regular hexagon to form a triangle, how many different triangles may be formed? (Two triangles are regarded to be the same if they are congruent.)

4. 若把兩個邊長分別爲 2 和 3 的正方體黏在一起，所得立體的表面積的最小可能值是甚麼？

When two cubes with side lengths 2 and 3 are stuck together, what is the minimum possible surface area of the resulting solid?

第 5 至第 8 題，每題 4 分。

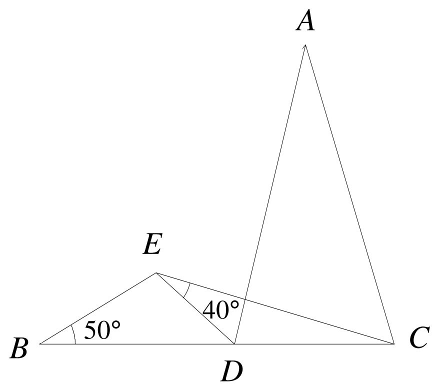
Questions 5 to 8 each carries 4 marks.

5. 設 m 是兩位正整數，m 的數字之和是 n 而 n 的數字之和是 8。問 m 有多少個不同的可能值？

Let m be a two-digit positive integer with sum of digits n. If the sum of digits of n is 8, how many different possible values of m are there?

6. 圖中，D 是 BC 上的一點，E 是 C 點沿直線 AD 反射的影像。若 $\angle DEC = 40°$ 而 $\angle EBC = 50°$，求 $\dfrac{BC}{DC}$。

In the figure, D is a point on BC and E is the image of C under reflection across the line AD. If $\angle DEC = 40°$ and $\angle EBC = 50°$, find $\dfrac{BC}{DC}$.



7. 子峰在黑板上以十進制寫下 2^{2009}。之後他算出黑板上的數的數字之和再乘以 9，然後把黑板上的數擦掉，寫上他的答案。假如子峰把這個過程重覆 2009 次，求黑板上最後出現的整數。

Alan wrote the number 2^{2009} on the board in decimal notation. He then computed the sum of digits of the number on the board and then multiplied by 9. After that he erased the number on the board and replaced it by his answer. If this process is repeated 2009 times, what would be the number that eventually appeared on the board?

8. 三個兩位正整數之積是 3240，而其中一個剛好等於三個數的平均數。求此平均數。

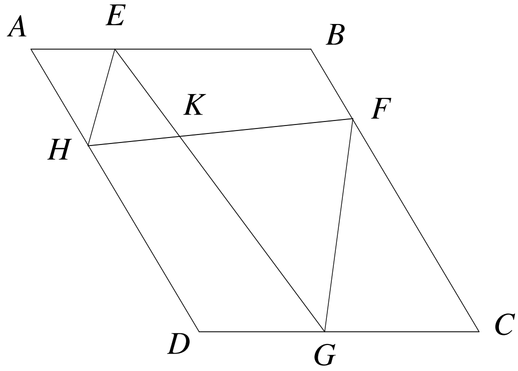
The product of three two-digit positive integers is 3240. One of the integers is equal to the mean of the three numbers. Find this mean.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 圖中，$ABCD$ 是邊長爲 17 的菱形。E、F、G、H 分別是 AB、BC、CD 和 DA 上的點，使得 $AE = 4$、$BF = 5$、$CG = 8$、$DH = 11$。設 EG 與 FH 交於 K。若 ΔKFG 的面積是 ΔKEH 的面積的 k 倍，求 k。

   In the figure, $ABCD$ is a rhombus with side length 17. E, F, G, H are points on AB, BC, CD and DA respectively such that $AE = 4$, $BF = 5$, $CG = 8$ and $DH = 11$. Let EG and FH intersect at K. If the area of ΔKFG is k times that of ΔKEH, find k.

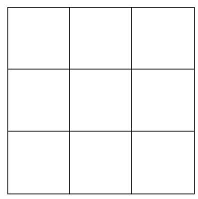


10. 現容許在整數 11111111 的兩個連續「1」字之間加上「+」號來組成不同的正整數，例如：我們可以得到 $1111+1111=2222$ 和 $1+11+1+1+11+1=26$。可以這樣得出來的正整數稱爲「好數」。求所有三位「好數」之和。

    From the number 11111111, one is allowed to insert the symbol '+' between two consecutive 1's to form various positive integers. For instance, one may get $1111+1111=2222$ and $1+11+1+1+11+1=26$. Positive integers which can be formed in this way are said to be 'good'. Find the sum of all 'good' three-digit numbers.

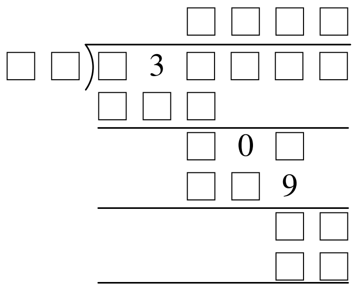
11. 汶蕙在圖中的 3×3 方格表裏每小格填進一個正整數（容許重覆）。填好後，她算算每直行和橫行的和，發現六個和都是互不相同的質數。若方格表裏的九個數之和是 m，求 m 的最小可能值。

    Cherry filled every cell of a 3×3 grid in the figure with a positive integer (repetition is allowed). After that, she computed the sum in each row and column and found that these six sums are pairwise distinct prime numbers. If the sum of the nine numbers in the grid is m, find the smallest possible value of m.



12. 圖中顯示一條除式，但當中有些數字留空了。求該六位的被除數。

The figure shows a division, but some digits are left out. Find the six-digit dividend.



第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 求 0 至 1 之間所有分母是 2009 的最簡分數之和。

Find the sum of all irreducible fractions between 0 and 1 whose denominator is equal to 2009.

14. 設 $[x]$ 代表不超過 x 的最大整數，例如 $[1.1] = 1$、$[6.9] = 6$ 和 $[5] = 5$。小儀選了一個正數 M，並發現只要 a、b 是正數且 $[a] \neq [b]$ 時，則 $\frac{a^2 - b^2}{[a]^2 - [b]^2}$ 的值總是小於 M。求 M 的最小可能值。

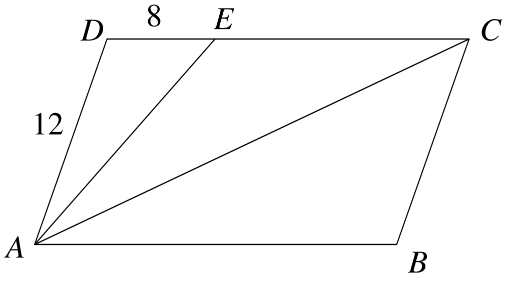
Let $[x]$ denote the greatest integer not exceeding x. For example, $[1.1] = 1$, $[6.9] = 6$ and $[5] = 5$. Sharon has chosen a positive number M and she finds that the value of $\frac{a^2 - b^2}{[a]^2 - [b]^2}$ is always less than M whenever a, b are positive numbers satisfying $[a] \neq [b]$. Find the smallest possible value of M.

15. 如果對任意正整數 n 皆有 $a_{n+k} = a_n$，則我們說 k 是無窮數列 $a_1, a_2, \ldots$ 的一個週期。例如：數列 1, 3, 4, 1, 3, 4, 1, 3, 4, … 的一個週期是 3。有多少個數列的每項皆是 1、2、3、4 或 5，且最小週期是 6？

If an infinite sequence $a_1, a_2, \ldots$ satisfies $a_{n+k} = a_n$ for all positive integers n, we say that k is a period of the sequence. For instance, 3 is a period of the sequence 1, 3, 4, 1, 3, 4, 1, 3, 4, … How many sequences have each term being 1, 2, 3, 4 or 5 and have smallest period 6?

16. 圖中，$ABCD$ 是平行四邊形，其中 $\angle CAD = 2\angle CAB$。$\angle CAD$ 的角平分線交 CD 於 E。若 $AD = 12$ 而 $DE = 8$，求 AC。

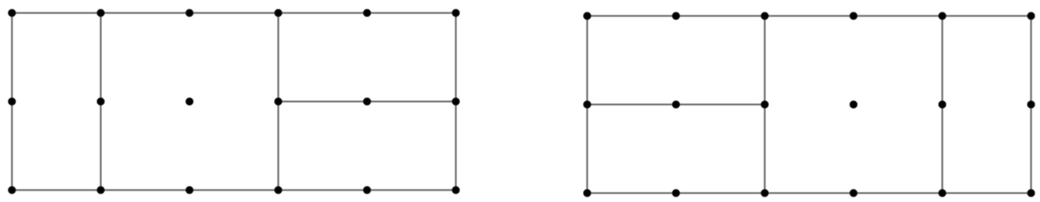
In the figure, $ABCD$ is a parallelogram with $\angle CAD = 2\angle CAB$. The bisector of $\angle CAD$ meets CD at E. If $AD = 12$ and $DE = 8$, find AC.



第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 俊華把一些 1×2 、2×1 和 2×2 大小的咭片砌成 $2\times n$ 大小的長方形（當中咭片不許重疊）。下圖顯示了兩個 2×5 大小的長方形，其中把左方的長方形繞其中心旋轉 180° 便可得到右方的長方形。若一個 $2\times n$ 大小的長方形繞其中心旋轉 180° 所得的影像與原圖形相同，那麼我們說它是「對稱」的。俊華有多少種方法砌出一個 2×12 的「對稱」長方形？



John uses some 1×2 , 2×1 and 2×2 cardboards to make up (without overlapping) $2\times n$ rectangles. Two 2×5 rectangles are shown above, with the one on the right obtained from the one on the left by a rotation of 180° about the centre of the figure. A $2\times n$ rectangle is said to be ‘symmetric’ if a rotation of 180° about its centre gives an image which is identical to the original rectangle. In how many different ways can John make a ‘symmetric’ 2×12 rectangle?

18. 若要把數字 1、3、3、8、8、8 重新排列成一個可被 7 整除的六位數，共有多少個不同的排列？

If the digits 1, 3, 3, 8, 8, 8 are to be permuted to form a six-digit number divisible by 7, how many different permutations are possible?

19. 已知 a、b、c、d、e、f、g 是七個互不相同的正整數，其中 $a<b$ 、 $a<c$ 而 $d<e<f$ 。小敏希望把七個數從小至大排列，她每次會選其中兩個數，然後小婷會告訴小敏兩個數中哪個較大。這個過程最少要進行多少次，小敏才可保證正確地排序？

Let a, b, c, d, e, f, g be seven pairwise different positive integers such that $a<b$, $a<c$ and $d<e<f$. Mandy wishes to arrange these numbers in ascending order. Each time she picks two of the numbers and Tiffany will tell her which of the two numbers is larger. At least how many times must this process be carried out in order that Mandy can ensure a correct arrangement?

20. 設 m、n 爲正整數。當 $\frac{m}{n}$ 寫成小數時，小數點後其中三個連續的數字依次是 1、6、7。求 n 的最小可能値。

Let m, n be positive integers. When $\frac{m}{n}$ is expressed as a decimal, three consecutive digits after the decimal point are 1, 6, 7 in order. Find the smallest possible value of n.

全卷完

END OF PAPER