第七屆培正數學邀請賽
7th Pui Ching Invitational Mathematics Competition

決賽（高中組）
Final Event (Senior Secondary)

時限：2 小時
Time allowed: 2 hours

參賽者須知：
Instructions to Contestants:

1. 本卷共設 20 題，總分爲 100 分。
   There are 20 questions in this paper and the total score is 100.

2. 除特別指明外，本卷內的所有數均爲十進制。
   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。
   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。
   Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。
   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。
   The diagrams in this paper are not necessarily drawn to scale.

1. 小娜打算用一些骰子來顯示她最近一次測驗的分數。已知測驗分數必定是小於 100 的正整數。當分數小於 10 時，則以 0 顯示十位（例如：8 分以「08」表示）。假設寫上「6」和「9」的面可以互相對調使用，而且每個面只可寫一個數字，那麼小娜最少要幾顆骰子？ （3 分）

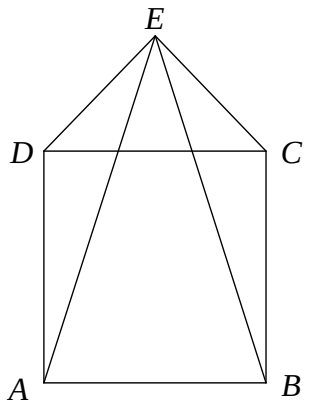
Ivana has some dice and she plans to use them to display her latest test score. It is known that each test score is a positive integer less than 100. When the score is less than 10, the tens digit will be displayed as '0'. (For example, a test score of 8 is displayed as '08'.) Assuming that faces marked '6' and '9' can be used interchangeably, and that only one digit may be written on each face, what is the minimum number of dice needed? (3 marks)

2. 若 $\int_{\frac{\pi}{2008}}^{\frac{1003\pi}{2008}} \frac{dx}{1+\tan x} = k\pi$ ，求 k。 （4 分）

If $\int_{\frac{\pi}{2008}}^{\frac{1003\pi}{2008}} \frac{dx}{1+\tan x} = k\pi$, find k. (4 marks)

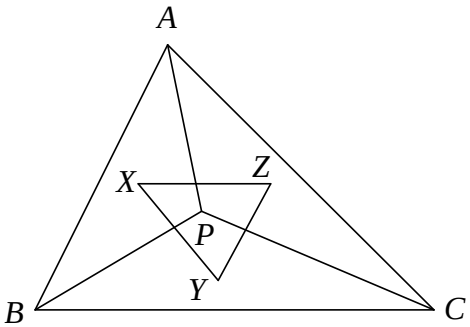
3. 圖中，$ABCD$ 是正方形、CDE 是直角等腰三角形（E 是直角）。求 $\sin\angle AEB$。 （4 分）

In the figure, $ABCD$ is a square while CDE is a right-angled isosceles triangle with right angle at E. Find $\sin\angle AEB$. (4 marks)



4. 圖中，P 是 ΔABC 內的一點，X、Y、Z 分別是 ΔPAB、ΔPBC 和 ΔPCA 的重心。若 $AB=15$ 、 $BC=39$ 、 $CA=36$ ，求 ΔXYZ 的周界。 （4 分）

In the figure, P is a point inside ΔABC and X, Y, Z are the centroids of ΔPAB, ΔPBC and ΔPCA respectively. If $AB=15$, $BC=39$ and $CA=36$, find the perimeter of ΔXYZ. (4 marks)



5. 求方程 $(\log x)^2 - \log x^2 - 1 = 0$ 所有實根之積。（4分）

Find the product of all real roots to the equation $(\log x)^2 - \log x^2 - 1 = 0$. (4 marks)

6. 在以下方程組中求 $x + y + z$ 的最大可能值。（5分）

Find the maximum possible value of $x + y + z$ in the following system of equations.

(5 marks)

$$\begin{cases} x^2 - (y - z)x - yz = 0 \\ y^2 - \left(\dfrac{8}{z^2} - x\right)y - \dfrac{8x}{z^2} = 0 \\ z^2 - (x - y)z - xy = 0 \end{cases}$$

7. 對於正整數 n，設 $f(n)$ 爲順序列出 1 至 n 所得的正整數，例如：$f(3) = 123$、$f(11) = 1234567891011$ 等；$g(n)$ 爲順序列出 $f(1)$ 至 $f(n)$ 所得的正整數，例如：$g(3) = 112123$、$g(6) = 112123123412345123456$ 等。求 $g(2008)$ 由左邊數起的第 2008 位數字。（5分）

For positive integer n, let $f(n)$ be the positive integer formed by listing 1 to n in order, e.g. $f(3) = 123$ and $f(11) = 1234567891011$; let also $g(n)$ be the positive integer formed by listing $f(1)$ to $f(n)$ in order, e.g. $g(3) = 112123$ and $g(6) = 112123123412345123456$. Find the 2008th digit of $g(2008)$ from the left. (5 marks)

8. 一位老師和 100 名學生參加一個遊戲，規則如下：主持人每次問一道題，答錯者被淘汰出局，答對者則可繼續回答下一題。若老師被淘汰或所有學生均被淘汰，遊戲便結束。每當老師答對一道問題，他皆可獲得獎金，數目是 1000 元乘以回答該題的學生中答錯的百分比。已知主持人問了三道題後遊戲便結束，而老師共獲得 n 元獎金，求 n 的最大可能值。（5分）

A teacher and 100 students join a game. The rule is as follows. One question is asked each time; those answering it wrongly are eliminated and the rest proceeds to answer the next question. The game ends when the teacher is eliminated or if all students are eliminated. Whenever the teacher answers a question correctly, he gets a bonus equal to 1000 dollars multiplied by the percentage of wrong answers among students answering that question. Given that the game ends after three questions are asked and the teacher gets a total bonus of n dollars, find the greatest possible value of n. (5 marks)

9. 設 p、q、r 爲方程 $x^3-12x^2+9x+3=0$ 的三個根。求下式的值：

$$p^2q+q^2r+r^2p+p^2r+r^2q+q^2p$$

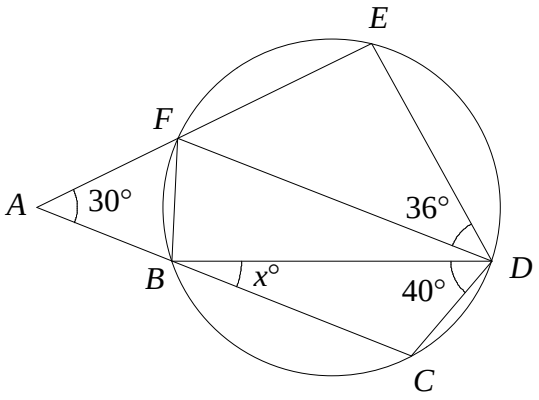
（5分）

Let p, q, r be the three roots of the equation $x^3-12x^2+9x+3=0$. Find the value of $p^2q+q^2r+r^2p+p^2r+r^2q+q^2p$. (5 marks)

10. 設 n 爲正整數。若 $\frac{n(n+1)(2n+1)}{6}$ 是 2008 的倍數，求 n 的最小可能值。（5分）

Let n be a positive integer. If $\frac{n(n+1)(2n+1)}{6}$ is a multiple of 2008, find the smallest possible value of n. (5 marks)

11. 圖中， $BCDEF$ 是圓內接五邊形，CB 與 EF 延長後交於 A。若 $\angle EAC = 30°$、$\angle EDF = 36°$、$\angle BDC = 40°$、$\angle DBC = x°$ 且 $\overset{\frown}{DE} = 3\overset{\frown}{DC}$，求 x。（5分）



In the figure, $BCDEF$ is a cyclic pentagon. CB and EF are produced to meet at A. If $\angle EAC = 30°$, $\angle EDF = 36°$, $\angle BDC = 40°$, $\angle DBC = x°$ and $\overset{\frown}{DE} = 3\overset{\frown}{DC}$, find x. (5 marks)

12. 某班共有 40 名學生，學號爲 1 至 40。他們測驗時都需要在試卷寫上自己的學號。某天測驗後收回試卷，班長把 40 份試卷上的學號加起來，發現它們之和是 k 的倍數，其中 k 是正整數。當老師得知 k 的值，並知道每位學生寫下的學號都是一個 1 至 40 當中的整數時，他立即斷定最少 2 名學生寫錯了學號。求 k 的最小可能值。（5分）

There are 40 students in a class, numbered 1 to 40. They were required to write their class numbers on the test paper during a test. After collecting the test papers after the test was over, the monitor added up the class numbers written on the 40 test papers, and discovered that the sum of the class numbers is a multiple of k, where k is a positive integer. When the teacher was told the value of k and the fact that every class number written was an integer from 1 to 40, he immediately deduced that at least 2 students had written a wrong class number. Find the least possible value of k. (5 marks)

13. 現有五名小朋友，分別編號爲 1、2、3、4、5。有多少種方法安排他們由左至右坐成一排，使得任意選出其中三位小朋友時，他們的編號（按左至右順序）都不會組成等差數列？ （6分）

Five children are numbered 1, 2, 3, 4, 5. In how many ways can the children be seated in a row from left to right so that if we choose any three children, their numbers read from left to right do not form an arithmetic sequence? (6 marks)

14. 求滿足以下兩個條件的 25 位正整數的數目：

- 該數的數字由剛好 20 個「1」和 5 個「2」組成。
- 如果把整數的最後 k 個數字刪去（其中 k 是任意小於 25 的正整數），則所得的新整數中「1」的數目不少於「2」的數目。 （6分）

Find the number of 25-digit positive integers satisfying the following two conditions:

- The digits of the integer consist of exactly 20 copies of '1' and 5 copies of '2'.
- If we remove the last k digits from the integer (where k is any positive integer less than 25), the resulting integer has at least as many '1's as '2's. (6 marks)

15. 若一個正整數可通過把另一正整數連續寫兩次或以上而得到，則此正整數稱爲「重複數」。例如：1212、88888、454545 都是「重複數」，7、1001、12344321 則不是「重複數」。求可被 273 整除的最小「重複數」。 （6分）

A positive integer is said to be 'repetitive' if it can be formed by repeatedly writing some other positive integer two or more times. For example, 1212, 88888, 454545 are 'repetitive' while 7, 1001, 12344321 are not. Find the smallest 'repetitive' number that is divisible by 273. (6 marks)

16. 考慮坐標平面上 (0,0)、(1,2)、(3,3)、(4,1) 和 A、B、C、D、E、F 共十點。已知在這十點中任選五點時，當中皆有四點共圓，且 A、B、C、D、E、F 六點中，任意兩點之間的距離均不小於 x。求 x 的最大可能值。 （7分）

Consider the ten points (0,0), (1,2), (3,3), (4,1) and A, B, C, D, E, F on the coordinate plane. It is known that if any five points are chosen from these ten points, there must exist four points which are concyclic. Furthermore, the distance between any two points among A, B, C, D, E, F is not less than x. Find the greatest possible value of x. (7 marks)

細閱以下資料，然後回答第 17 和第 18 題。

Study the following information and answer Questions 17 and 18.

正整數可以羅馬數字來表示。羅馬數字由七個字母 I、V、X、L、C、D、M 組成。下表顯示小於 3000 的正整數的羅馬數字表示式，例如：2008 的表示式爲 MMVIII、1988 爲 MCMLXXXVIII、700 則爲 DCC。若某整數的表示式上下倒轉看時仍代表某整數，則原來的整數稱爲「好數」。例如：XXX 倒轉看時仍是「XXX」，所以 30 是「好數」；XXI 倒轉看時變成「IXX」，不是某整數的羅馬數字表示式，所以 21 不是「好數」；IV 倒轉看時變成「ΛI」，所以 4 也不是「好數」。

Positive integers can be represented by Roman numerals. Roman numerals are made up of the seven letters I, V, X, L, C, D, M. The table below lists the Roman numeral representations of positive integers less than 3000. For instance, 2008 is written as MMVIII, 1988 is written as MCMLXXXVIII while 700 is written as DCC. A positive integer is said to be 'good' if its Roman numeral representation is still the representation of some integer when read upside down. For instance, 30 is 'good' since XXX still reads 'XXX' when upside down; 21 is not 'good' since XXI reads 'IXX' when upside down, not representing any integer; and 4 is not 'good' since IV reads 'ΛI' when upside down.

	1	2	3	4	5	6	7	8	9
千位 Thousands	M	MM							
百位 Hundreds	C	CC	CCC	CD	D	DC	DCC	DCCC	CM
十位 Tens	X	XX	XXX	XL	L	LX	LXX	LXXX	XC
個位 Unit	I	II	III	IV	V	VI	VII	VIII	IX

17. 在首 2008 個正整數中，有多少個是「好數」？ （4 分）

How many of the first 2008 positive integers are 'good'? (4 marks)

18. 在首 2008 個正整數中，有多少個的羅馬數字表示式剛好有 8 個字母？ （6 分）

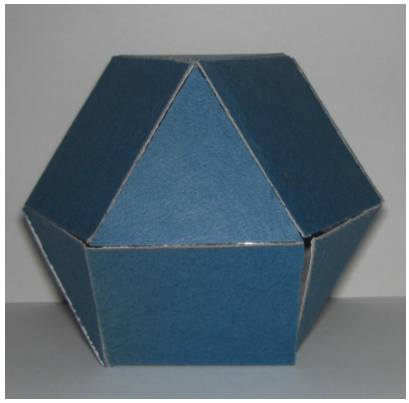
How many of the first 2008 positive integers have their Roman numeral representation consisting of exactly 8 letters? (6 marks)

細閱以下資料，然後回答第 19 和第 20 題。

Study the following information and answer Questions 19 and 20.

圖中的立體有 14 個面，其中包括 6 個正方形和 8 個正三角形（頂和底都是三角形）。立體每邊的長度都是 1，每個正方形面都和 4 個三角形面相鄰，而每個三角形面都和 3 個正方形面相鄰。

The solid in the figure consists of 14 faces, including 6 squares and 8 equilateral triangles (both the top and bottom faces are triangles). Each edge of the solid has length 1, each square face is adjacent to 4 triangular faces and each triangular face is adjacent to 3 square faces.



19. 這個立體可以製成一顆骰子。現於八個三角形面上分別寫上 1 至 8、六個正方形面上分別寫上 9 至 14。已知擲得某面的概率跟其面積成正比，則投擲這顆骰子時擲出質數的概率是多少？ （5 分）

    A die can be made from this solid. We write the numbers 1 to 8 on the eight triangular faces and the numbers 9 to 14 on the six square faces. Given that the probability of obtaining a particular face is directly proportional to its area, what is the probability that a prime number is obtained when the die is thrown? (5 marks)

20. 設立體中兩個相鄰面之間的鈍角爲 θ。求 $\tan\theta$。 （6 分）

    Let θ be the obtuse angle between two adjacent faces of the solid. Find $\tan\theta$. (6 marks)

全卷完

END OF PAPER