第七屆培正數學邀請賽
7th Pui Ching Invitational Mathematics Competition

決賽（中四組）
Final Event (Secondary 4)

時限：2 小時
Time allowed: 2 hours

參賽者須知：
Instructions to Contestants:

1. 本卷共設 20 題，總分爲 100 分。
   There are 20 questions in this paper and the total score is 100.

2. 除特別指明外，本卷內的所有數均爲十進制。
   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。
   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案塡在答題紙指定的空位上。毋須呈交計算步驟。
   Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。
   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。
   The diagrams in this paper are not necessarily drawn to scale.

1. 若一個正整數可通過把另一正整數連續寫兩次或以上而得到，則此正整數稱爲「重複數」。例如：1212、88888、454545 都是「重複數」，7、1001、12344321 則不是「重複數」。若 n 是正整數而 $10n$ 是「重複數」，求 n 的最小可能值。（3分）

A positive integer is said to be 'repetitive' if it can be formed by repeatedly writing some other positive integer two or more times. For example, 1212, 88888, 454545 are 'repetitive' while 7, 1001, 12344321 are not. If n is a positive integer for which $10n$ is 'repetitive', find the smallest possible value of n. (3 marks)

2. 20080308 名參賽者排成一列。現在按以下的方法淘汰參賽者：首先，從隊首將他們順序由 1 開始以整數順序編號，把所有偶數號的參賽者淘汰。然後，把未被淘汰的人按剛才的隊伍順序排列，重新由 1 開始以整數順序編號，把所有編號並非 5 的倍數的參賽者淘汰。如是者，不斷重覆上述的步驟，梅花間竹地淘汰編號是偶數和並非 5 的倍數的參賽者，直至只餘下一名參賽者爲止。最後餘下的參賽者第一次排隊時的編號是甚麼？（4分）

20080308 contestants align in a row. Some contestants are eliminated with the following procedures. First number each contestant with integers starting from 1 in ascending order, and all contestants with even numbers are eliminated. The remaining contestants are then aligned in the previous order and re-numbered with integers from 1 onward again. Contestants whose numbers are not multiples of 5 are eliminated. The above procedures are repeated, alternately eliminating contestants with even numbers and with numbers not divisible by 5, until only one contestant remains. What is the number of the remaining contestant during the first round? (4 marks)

3. 設 $[x]$ 代表不超過 x 的最大整數，例如 $[1.1] = 1$、$[6.9] = 6$ 和 $[5] = 5$。求

$$\left[\frac{2008}{1}\right]-\left[\frac{2007}{1}\right]+\left[\frac{2008}{2}\right]-\left[\frac{2007}{2}\right]+\left[\frac{2008}{3}\right]-\left[\frac{2007}{3}\right]+\cdots+\left[\frac{2008}{2008}\right]-\left[\frac{2007}{2008}\right]$$

的值。（4分）

Let $[x]$ denote the greatest integer not exceeding x. For example, $[1.1] = 1$, $[6.9] = 6$ and $[5] = 5$. Find the value of

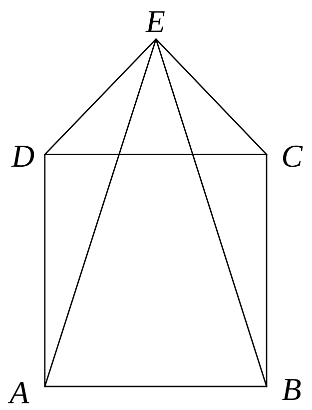
$$\left[\frac{2008}{1}\right]-\left[\frac{2007}{1}\right]+\left[\frac{2008}{2}\right]-\left[\frac{2007}{2}\right]+\left[\frac{2008}{3}\right]-\left[\frac{2007}{3}\right]+\cdots+\left[\frac{2008}{2008}\right]-\left[\frac{2007}{2008}\right].$$ (4 marks)

4. 若 $a_1=\frac{1}{6}$，且對 $n>1$ 皆有 $a_n=a_{n-1}-\frac{2}{n(n+1)(n+2)}$，求 a_{100}。 （4分）

Suppose $a_1=\frac{1}{6}$ and $a_n=a_{n-1}-\frac{2}{n(n+1)(n+2)}$ for $n>1$. Find a_{100}. (4 marks)

5. 圖中，*ABCD* 是正方形、*CDE* 是直角等腰三角形（*E* 是直角）。求 sin ∠*AEB*。 （4分）

In the figure, *ABCD* is a square while *CDE* is a right-angled isosceles triangle with right angle at *E*. Find sin ∠*AEB*. (4 marks)

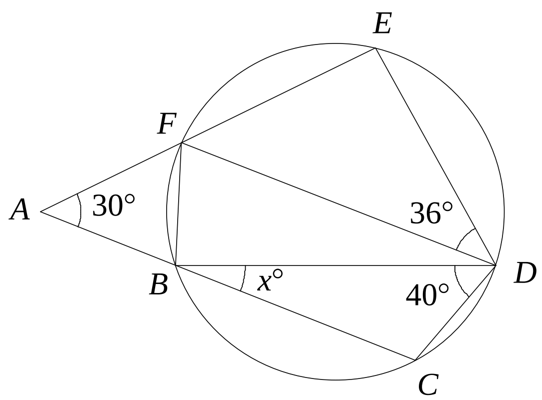


6. 考慮坐標平面上共圓的四點 *A*、*B*、*C*、*D*，其中 *B* = (17,3)、*C* = (9,–9)、*D* = (–1,15)。若 *AB*、*AC* 和 *AD* 三個線段中最短的一個長度爲 *x*，求 *x* 的最大可能值。 （5分）

Consider the four concyclic points *A*, *B*, *C*, *D* on the coordinate plane, where *B* = (17,3), *C* = (9,–9) and *D* = (–1,15). If the shortest of the lengths of the segments *AB*, *AC* and *AD* is *x*, find the greatest possible value of *x*. (5 marks)

7. 圖中， *BCDEF* 是圓內接五邊形，*CB* 與 *EF* 延長後交於 *A*。若 ∠*EAC* = 30°、∠*EDF* = 36°、∠*BDC* = 40°、∠*DBC* = *x*° 且 $\overset{\frown}{DE}=3\overset{\frown}{DC}$，求 *x*。 （5分）

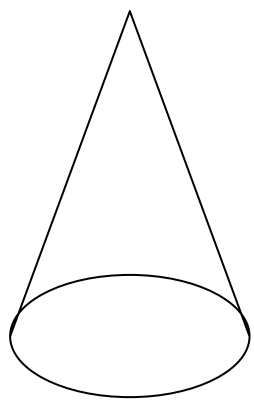
In the figure, *BCDEF* is a cyclic pentagon. *CB* and *EF* are produced to meet at *A*. If ∠*EAC* = 30°, ∠*EDF* = 36°, ∠*BDC* = 40°, ∠*DBC* = *x*° and $\overset{\frown}{DE}=3\overset{\frown}{DC}$, find *x*. (5 marks)



8. 已知 m 爲實數。若方程 $x^4-20x^2+m=0$ 的四個實數解是某等差數列的連續四項，求 m 的值。（5 分）

It is known that m is a real number. If the four real roots of the equation $x^4-20x^2+m=0$ are four consecutive terms of an arithmetic sequence, find the value of m. (5 marks)

9. 一個放在地面上的直立圓錐型容器裏盛滿了水。如果在圓錐的側面開小洞，小洞的位置以上的水便會從各小洞以相同的均速流失，直至水面降至洞的水平以下。假如在離地面 10 米處開一個小洞，水流會在開洞後 54 分鐘停止。假如同時在離地面 10 米和 12 米處各開一個小洞，則水流會在開洞後 46 分鐘停止。圓錐的高度是多少米？（5 分）



An upright conical container placed on the ground is full of water. If holes are drilled on the side of the cone, water above the cone will flow away from each hole at the same uniform speed until the water level falls below the hole. If a hole is drilled 10 m above the ground, water flow will stop 54 minutes after the drill. If one hole is drilled at each of 10 m and 12 m above the ground at the same time, water flow will stop 46 minutes after the drill. Find the height of the cone (in metres). (5 marks)

10. 一位老師和 100 名學生參加一個遊戲，規則如下：主持人每次問一道題，答錯者被淘汰出局，答對者則可繼續回答下一題。若老師被淘汰或所有學生均被淘汰，遊戲便結束。每當老師答對一道問題，他皆可獲得獎金，數目是 1000 元乘以回答該題的學生中答錯的百分比。已知主持人問了三道題後遊戲便結束，而老師共獲得 n 元獎金，求 n 的最大可能值。（5 分）

A teacher and 100 students join a game. The rule is as follows. One question is asked each time; those answering it wrongly are eliminated and the rest proceeds to answer the next question. The game ends when the teacher is eliminated or if all students are eliminated. Whenever the teacher answers a question correctly, he gets a bonus equal to 1000 dollars multiplied by the percentage of wrong answers among students answering that question. Given that the game ends after three questions are asked and the teacher gets a total bonus of n dollars, find the greatest possible value of n. (5 marks)

11. 若要從 1、2、⋯、100 中選出兩個不同的數，使得當中既有 4 的倍數，亦有 6 的倍數，問共有多少種不同的選法？ （5 分）

In how many different ways can we choose two different numbers from 1, 2, …, 100 so that there exists a multiple of 4 as well as a multiple of 6 among the chosen numbers? (5 marks)

12. 若 $\begin{cases} 2\sin^2 A-2\tan A\sin A+2-\sin^2 A-\cos^2 B=0 \\ 2\cos^2 B-2\tan A\cos B+2-\sin^2 A-\cos^2 B=0 \end{cases}$，求 $\sin^2 A$ 所有可能值之和，答案以 $a+\sqrt{b}$ 或 $a-\sqrt{b}$ 形式表示，其中 a 和 b 爲有理數。 （6 分）

If $\begin{cases} 2\sin^2 A-2\tan A\sin A+2-\sin^2 A-\cos^2 B=0 \\ 2\cos^2 B-2\tan A\cos B+2-\sin^2 A-\cos^2 B=0 \end{cases}$, find the sum of all possible values of $\sin^2 A$ in the form $a+\sqrt{b}$ or $a-\sqrt{b}$, where a, b are rational numbers. (6 marks)

13. 現有五名小朋友，分別編號爲 1、2、3、4、5。有多少種方法安排他們由左至右坐成一排，使得任意選出其中三位小朋友時，他們的編號（按左至右順序）都不會組成等差數列？ （6 分）

Five children are numbered 1, 2, 3, 4, 5. In how many ways can the children be seated in a row from left to right so that if we choose any three children, their numbers read from left to right do not form an arithmetic sequence? (6 marks)

14. 求滿足以下兩個條件的 25 位正整數的數目：

- 該數的數字由剛好 20 個「1」和 5 個「2」組成。
- 如果把整數的最後 k 個數字刪去（其中 k 是任意小於 25 的正整數），則所得的新整數中「1」的數目不少於「2」的數目。 （6 分）

Find the number of 25-digit positive integers satisfying the following two conditions:

- The digits of the integer consist of exactly 20 copies of ‘1’ and 5 copies of ‘2’.
- If we remove the last k digits from the integer (where k is any positive integer less than 25), the resulting integer has at least as many ‘1’s as ‘2’s. (6 marks)

15. 某直角三角形各邊的長度（以厘米爲單位）皆是整數，其周界爲 k 厘米、面積爲 $2k$ 平方厘米。求 k 所有可能値之和。（6 分）

The length (in cm) of each side of a right-angled triangle is an integer. If its perimeter is k cm and its area is $2k$ cm^2, find the sum of all possible values of k. (6 marks)

16. 對於正整數 n，設 $f(n)$ 爲順序列出 1 至 n 所得的正整數，例如：$f(3)=123$、$f(11)=1234567891011$ 等；$g(n)$ 爲順序列出 $f(1)$ 至 $f(n)$ 所得的正整數，例如：$g(3)=112123$、$g(6)=112123123412345123456$ 等。求 $g(20)$ 除以 99 時的餘數。（7 分）

For positive integer n, let $f(n)$ be the positive integer formed by listing 1 to n in order, e.g. $f(3)=123$ and $f(11)=1234567891011$; let also $g(n)$ be the positive integer formed by listing $f(1)$ to $f(n)$ in order, e.g. $g(3)=112123$ and $g(6)=112123123412345123456$. Find the remainder when $g(20)$ is divided by 99. (7 marks)

細閱以下資料，然後回答第 17 和第 18 題。

Study the following information and answer Questions 17 and 18.

正整數可以羅馬數字來表示。羅馬數字由七個字母 I、V、X、L、C、D、M 組成。下表顯示小於 3000 的正整數的羅馬數字表示式，例如：2008 的表示式爲 MMVIII、1988 爲 MCMLXXXVIII、700 則爲 DCC。若某整數的表示式上下倒轉看時仍代表某整數，則原來的整數稱爲「好數」。例如：XXX 倒轉看時仍是「XXX」，所以 30 是「好數」；XXI 倒轉看時變成「IXX」，不是某整數的羅馬數字表示式，所以 21 不是「好數」；IV 倒轉看時變成「ΛI」，所以 4 也不是「好數」。

Positive integers can be represented by Roman numerals. Roman numerals are made up of the seven letters I, V, X, L, C, D, M. The table below lists the Roman numeral representations of positive integers less than 3000. For instance, 2008 is written as MMVIII, 1988 is written as MCMLXXXVIII while 700 is written as DCC. A positive integer is said to be ‘good’ if its Roman numeral representation is still the representation of some integer when read upside down. For instance, 30 is ‘good’ since XXX still reads ‘XXX’ when upside down; 21 is not ‘good’ since XXI reads ‘IXX’ when upside down, not representing any integer; and 4 is not ‘good’ since IV reads ‘ΛI’ when upside down.

	1	2	3	4	5	6	7	8	9
千位 Thousands	M	MM							
百位 Hundreds	C	CC	CCC	CD	D	DC	DCC	DCCC	CM
十位 Tens	X	XX	XXX	XL	L	LX	LXX	LXXX	XC
個位 Unit	I	II	III	IV	V	VI	VII	VIII	IX

17. 在首 2008 個正整數中，有多少個是「好數」？ （4 分）

    How many of the first 2008 positive integers are ‘good’? (4 marks)

18. 在首 2008 個正整數中，有多少個的羅馬數字表示式包含字母「C」？ （5 分）

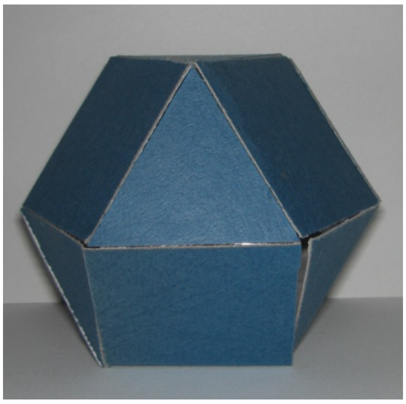
    How many of the first 2008 positive integers have their Roman numeral representation consisting of the letter ‘C’? (5 marks)

細閱以下資料，然後回答第 19 和第 20 題。

Study the following information and answer Questions 19 and 20.

圖中的立體有 14 個面，其中包括 6 個正方形和 8 個正三角形（頂和底都是三角形）。立體每邊的長度都是 1，每個正方形面都和 4 個三角形面相鄰，而每個三角形面都和 3 個正方形面相鄰。

The solid in the figure consists of 14 faces, including 6 squares and 8 equilateral triangles (both the top and bottom faces are triangles). Each edge of the solid has length 1, each square face is adjacent to 4 triangular faces and each triangular face is adjacent to 3 square faces.



19. 這個立體可以製成一顆骰子。現於八個三角形面上分別寫上 1 至 8、六個正方形面上分別寫上 9 至 14。已知擲得某面的概率跟其面積成正比，則投擲這顆骰子時擲出質數的概率是多少？ （5 分）

A die can be made from this solid. We write the numbers 1 to 8 on the eight triangular faces and the numbers 9 to 14 on the six square faces. Given that the probability of obtaining a particular face is directly proportional to its area, what is the probability that a prime number is obtained when the die is thrown? (5 marks)

20. 求立體的體積。 （6 分）

Find the volume of the solid. (6 marks)

全卷完

END OF PAPER