第七屆培正數學邀請賽
7th Pui Ching Invitational Mathematics Competition

決賽（中三組）
Final Event (Secondary 3)

時限：2 小時
Time allowed: 2 hours

參賽者須知：
Instructions to Contestants:

1. 本卷共設 20 題，總分爲 100 分。

    There are 20 questions in this paper and the total score is 100.

2. 除特別指明外，本卷內的所有數均爲十進制。

    Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

    Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

    Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。

    The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。

    The diagrams in this paper are not necessarily drawn to scale.

1. 一個遊戲中有 15 張分別編號爲 1 至 15 的咭片。參加者需要猜出每張咭片上的編號，而所猜的 15 個編號必須是 1 至 15 之間的不同整數。參加者的得分是 15 個猜測的絕對誤差之和。求得分的最大可能值。 （3分）

   In a game there are 15 cards, numbered 1 to 15. A player has to guess the number on each card, and the 15 guesses must be different integers from 1 to 15. The score of a player is the sum of the absolute errors of all 15 guesses. Find the maximum possible score. (3 marks)

2. 已知 n 是四位奇數，m 是大於 1 的立方數，且 n 是 m 的倍數。求 n 的最小可能值。 （3分）

   Given n is a four-digit odd number, m is a cubic number greater than 1 and n is a multiple of m. Find the smallest possible value of n. (3 marks)

3. 若一個正整數可通過把另一正整數連續寫兩次或以上而得到，則此正整數稱爲「重複數」。例如：1212、88888、454545 都是「重複數」，7、1001、12344321 則不是「重複數」。若 n 是所有四位「重複數」的因數，求 n 的最大可能值。 （4分）

   A positive integer is said to be 'repetitive' if it can be formed by repeatedly writing some other positive integer two or more times. For example, 1212, 88888, 454545 are 'repetitive' while 7, 1001, 12344321 are not. If n is a factor of all four-digit 'repetitive' numbers, find the greatest possible value of n. (4 marks)

4. 設 $a_1 = 1$、$a_2 = \dfrac{5}{2}$，且對任意大於 1 的整數 n 皆有 $a_{n+1} = a_n + \dfrac{2a_{n-1}}{n-1}$。求 a_{100}。 （4分）
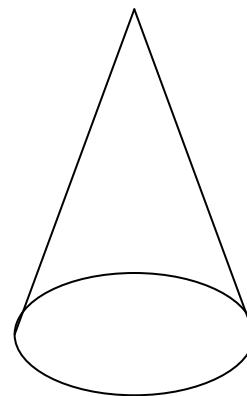
   Let $a_1 = 1$, $a_2 = \dfrac{5}{2}$ and $a_{n+1} = a_n + \dfrac{2a_{n-1}}{n-1}$ for all integers n greater than 1. Find a_{100}. (4 marks)

5. 圖中，P 是 ΔABC 內的一點，X、Y、Z 分別是 ΔPAB、ΔPBC 和 ΔPCA 的重心。若 $AB = 15$、$BC = 39$、$CA = 36$，求 ΔXYZ 的周界。 （4分）

   In the figure, P is a point inside ΔABC and X, Y, Z are the centroids of ΔPAB, ΔPBC and ΔPCA respectively. If $AB = 15$, $BC = 39$ and $CA = 36$, find the perimeter of ΔXYZ. (4 marks)
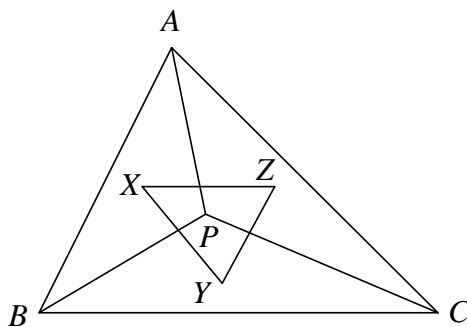
6. 一個放在地面上的直立圓錐型容器裏盛滿了水。如果在圓錐的側面開小洞，小洞的位置以上的水便會從各小洞以相同的均速流失，直至水面降至洞的水平以下。假如在離地面 10 米處開一個小洞，水流會在開洞後 54 分鐘停止。假如同時在離地面 10 米和 12 米處各開一個小洞，則水流會在開洞後 46 分鐘停止。圓錐的高度是多少米？ （5分）

An upright conical container placed on the ground is full of water. If holes are drilled on the side of the cone, water above the cone will flow away from each hole at the same uniform speed until the water level falls below the hole. If a hole is drilled 10 m above the ground, water flow will stop 54 minutes after the drill. If one hole is drilled at each of 10 m and 12 m above the ground at the same time, water flow will stop 46 minutes after the drill. Find the height of the cone (in metres). (5 marks)

7. 若要從 1、2、…、100 中選出兩個不同的數，使得其中一個可被 4 整除，另一個可被 6 整除，問共有多少種不同的選法？ （5分）

In how many different ways can we choose two different numbers from 1, 2, …, 100 so that one of the chosen numbers is divisible by 4 and the other is divisible by 6? (5 marks)

8. 設 n 為正整數。若 $\dfrac{n(n+1)(2n+1)}{6}$ 是 2008 的倍數，求 n 的最小可能值。 （5分）

Let n be a positive integer. If $\dfrac{n(n+1)(2n+1)}{6}$ is a multiple of 2008, find the smallest possible value of n. (5 marks)

9. 若 $\begin{cases} 2\sin^2 A - 2\tan A \sin A + 2 - \sin^2 A - \cos^2 B = 0 \\ 2\cos^2 B - 2\tan A \cos B + 2 - \sin^2 A - \cos^2 B = 0 \end{cases}$ 而 $\sin A \neq \cos B$，求 $\sin^2 A$ 的值。 （6分）

If $\begin{cases} 2\sin^2 A - 2\tan A \sin A + 2 - \sin^2 A - \cos^2 B = 0 \\ 2\cos^2 B - 2\tan A \cos B + 2 - \sin^2 A - \cos^2 B = 0 \end{cases}$ and $\sin A \neq \cos B$, find the value of $\sin^2 A$. (6 marks)

10. 在所示的算式中，不同的字母代表不同的數字。
已知 U = 6，求 CHING 代表的五位數。
（6分）

In the calculation shown, different letters represent different digits. Given that U = 6, find the five-digit number represented by CHING.

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    M  A  T  H
          A  T
+      P  U  I
-------------
 C  H  I  N  G
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(6 marks)

11. 下圖中有一條長 200 m 的軌道，軌道的兩端各有一塊板和一個球。開始時，球 A 以 x m/s 的速度向右走，而球 B 以 5 m/s 的速度向左走。每當兩個球相撞時，它們均即時轉向反方向走，且速度減低三分之二；每當球與板相撞時，球亦即時轉向反方向走，但速度增加 50%。若球 A 走了 2200 m 後其速度為 1 m/s，求 x。（球的半徑可略去不計。）
（6分）

In the above figure, there is a track of length 200 m. At each of the two ends of the track there is a barrier and a ball. Initially, ball A moves to the right at x m/s while ball B moves to the left at 5 m/s. Whenever the two balls collide, they immediately move in the opposite direction with speeds reduced by two-thirds; whenever a ball hits the barrier, it also turns immediately to move in the opposite direction but with the speed increased by 50%. If the speed of ball A is 1 m/s after it has travelled a total distance of 2200 m, find x. (The radius of the ball is negligible.)

(6 marks)

12. 如果一個四位數的每位數字皆不是 0，而四個數字之積是 9 的倍數，那麼這個四位數稱為「九倍數」。例如，因為 $9 \times 1 \times 3 \times 5 = 135$ 是 9 的倍數，故 9135 是「九倍數」；而 $2 \times 8 \times 3 \times 1 = 48$ 不是 9 的倍數，故 2831 不是「九倍數」。問共有多少個「九倍數」？
（6分）

A four-digit number is called a 'multi-9 number' if none of its digits is 0 and the product of the 4 digits is a multiple of 9. For example, since $9 \times 1 \times 3 \times 5 = 135$ is a multiple of 9, 9135 is a 'multi-9 number'; since $2 \times 8 \times 3 \times 1 = 48$ is not a multiple of 9, 2831 is not a 'multi-9 number'. How many 'multi-9 numbers' are there?
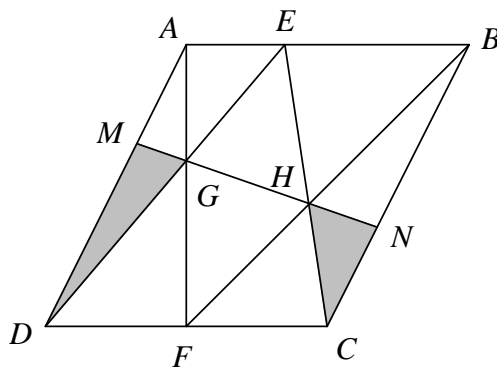
(6 marks)

13. 若要從一個正 20 邊形中選出一對互相平行的對角線，共有多少種不同的選法？ （6分）

In how many different ways can a pair of parallel diagonals be chosen in a regular 20-sided polygon? (6 marks)

14. 圖中， $ABCD$ 是平行四邊形，E 是 AB 上的一點使得 $AE:EB=1:2$，F 是 CD 的中點，AF 交 DE 於 G，BF 交 CE 於 H。GH 向兩端延長後分別與 AD 和 BC 相交於 M 和 N。若 $ABCD$ 的面積為 420，求陰影部分的面積之和。

In the figure, *ABCD* is a parallelogram. *E* is a point on *AB* such that $AE:EB=1:2$, *F* is the mid-point of *CD*, *AF* meets *DE* at *G* and *BF* meets *CE* at *H*. When produced on both sides, *GH* meets *AD* at *M* and *BC* at *N*. If the area of *ABCD* is 420, find the sum of the shaded areas.

（6分）

(6 marks)

15. 某直角三角形各邊的長度（以厘米為單位）皆是整數，其周界為 k 厘米、面積為 $2k$ 平方厘米。求 k 所有可能值之和。 （6分）

The length (in cm) of each side of a right-angled triangle is an integer. If its perimeter is *k* cm and its area is $2k$ cm^2, find the sum of all possible values of *k*. (6 marks)

16. 對於正整數 n，設 $f(n)$ 為順序列出 1 至 n 所得的正整數，例如：$f(3)=123$、$f(11)=1234567891011$ 等；$g(n)$ 為順序列出 $f(1)$ 至 $f(n)$ 所得的正整數，例如：$g(3)=112123$、$g(6)=112123123412345123456$ 等。求 $g(20)$ 除以 99 時的餘數。 （7分）

For positive integer *n*, let $f(n)$ be the positive integer formed by listing 1 to *n* in order, e.g. $f(3)=123$ and $f(11)=1234567891011$; let also $g(n)$ be the positive integer formed by listing $f(1)$ to $f(n)$ in order, e.g. $g(3)=112123$ and $g(6)=112123123412345123456$. Find the remainder when $g(20)$ is divided by 99. (7 marks)

細閱以下資料，然後回答第 17 和第 18 題。

Study the following information and answer Questions 17 and 18.

正整數可以羅馬數字來表示。羅馬數字由七個字母 I、V、X、L、C、D、M 組成。下表顯示小於 3000 的正整數的羅馬數字表示式，例如：2008 的表示式為 MMVIII、1988 為 MCMLXXXVIII、700 則為 DCC。若某整數的表示式上下倒轉看時仍代表某整數，則原來的整數稱為「好數」。例如：XXX 倒轉看時仍是「XXX」，所以 30 是「好數」；XXI 倒轉看時變成「IXX」，不是某整數的羅馬數字表示式，所以 21 不是「好數」；IV 倒轉看時變成「ΛI」，所以 4 也不是「好數」。

Positive integers can be represented by Roman numerals. Roman numerals are made up of the seven letters I, V, X, L, C, D, M. The table below lists the Roman numeral representations of positive integers less than 3000. For instance, 2008 is written as MMVIII, 1988 is written as MCMLXXXVIII while 700 is written as DCC. A positive integer is said to be 'good' if its Roman numeral representation is still the representation of some integer when read upside down. For instance, 30 is 'good' since XXX still reads 'XXX' when upside down; 21 is not 'good' since XXI reads 'IXX' when upside down, not representing any integer; and 4 is not 'good' since IV reads 'ΛI' when upside down.

	1	2	3	4	5	6	7	8	9
千位 Thousands	M	MM							
百位 Hundreds	C	CC	CCC	CD	D	DC	DCC	DCCC	CM
十位 Tens	X	XX	XXX	XL	L	LX	LXX	LXXX	XC
個位 Unit	I	II	III	IV	V	VI	VII	VIII	IX

17. 某「好數」m 的羅馬數字表示式上下倒轉看時變成了整數 n 的羅馬數字表示式。求 m 和 n 之差的最大可能值。 （3分）

    When read upside down, the Roman numeral representation of a 'good' integer m becomes the Roman numeral representation of the integer n. What is the maximum possible value of the difference between m and n? (3 marks)

18. 在首 2008 個正整數中，有多少個的羅馬數字表示式包含全部 7 個字母？ （5分）

    How many of the first 2008 positive integers have their Roman numeral representation consisting of all 7 letters? (5 marks)

細閱以下資料，然後回答第 **19** 和第 **20** 題。

Study the following information and answer Questions 19 and 20.

圖中的立體有 14 個面，其中包括 6 個正方形和 8 個正三角形（頂和底都是三角形）。立體每邊的長度都是 1，每個正方形面都和 4 個三角形面相鄰，而每個三角形面都和 3 個正方形面相鄰。

The solid in the figure consists of 14 faces, including 6 squares and 8 equilateral triangles (both the top and bottom faces are triangles). Each edge of the solid has length 1, each square face is adjacent to 4 triangular faces and each triangular face is adjacent to 3 square faces.
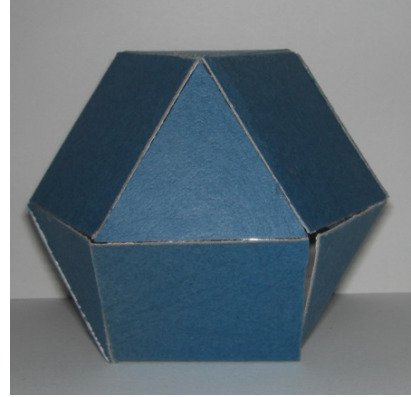
19. 若要從立體中選取兩個相鄰的面，共有多少種選法？ （4 分）

    In how many ways can we choose two adjacent faces from the solid? (4 marks)

20. 求立體的體積。 （6 分）

    Find the volume of the solid. (6 marks)

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END OF PAPER

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