第七屆培正數學邀請賽
7th Pui Ching Invitational Mathematics Competition

決賽（中二組）
Final Event (Secondary 2)

時限：2 小時
Time allowed: 2 hours

參賽者須知：
Instructions to Contestants:

1. 本卷共設 20 題，總分為 100 分。
   There are 20 questions in this paper and the total score is 100.

2. 除特別指明外，本卷內的所有數均為十進制。
   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。
   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。
   Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。
   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。
   The diagrams in this paper are not necessarily drawn to scale.

1. 平面上畫有 10 條無限長的直線，它們共構成 n 個交點，其中 $n>1$。求 n 的最小可能值。 （3分）

   On the plane 10 straight lines of infinite length are drawn. They produce a total of n points of intersection, where $n>1$. Find the smallest possible value of n. (3 marks)

2. 一個遊戲中有 15 張分別編號爲 1 至 15 的咭片。參加者需要猜出每張咭片上的編號，而所猜的 15 個編號必須是 1 至 15 之間的不同整數。參加者的得分是 15 個猜測的絕對誤差之和。求得分的最大可能值。 （3分）

   In a game there are 15 cards, numbered 1 to 15. A player has to guess the number on each card, and the 15 guesses must be different integers from 1 to 15. The score of a player is the sum of the absolute errors of all 15 guesses. Find the maximum possible score. (3 marks)

3. 下圖顯示以紙牌砌成的「紙牌金字塔」。如圖所示，砌出一層、兩層和三層的金字塔分別需要 2 張、7 張和 15 張紙牌。若依此規律砌出一個 100 層的金字塔，共需要多少張紙牌？ （3分）

   The above figure shows some 'card pyramids'. As shown in the figure, a pyramid with 1, 2 and 3 layers requires 2, 7 and 15 cards respectively. If we follow the pattern to build a card pyramid with 100 layers, how many cards are needed? (3 marks)

4. 對於正整數 n，設 $f(n)$ 爲順序列出 1 至 n 所得的正整數，例如：$f(3)=123$、$f(11)=1234567891011$ 等；$g(n)$ 爲順序列出 $f(1)$ 至 $f(n)$ 所得的正整數，例如：$g(3)=112123$、$g(6)=112123123412345123456$ 等。問 $g(64)$ 有多少個數字？ （4分）

   For positive integer n, let $f(n)$ be the positive integer formed by listing 1 to n in order, e.g. $f(3)=123$ and $f(11)=1234567891011$; let also $g(n)$ be the positive integer formed by listing $f(1)$ to $f(n)$ in order, e.g. $g(3)=112123$ and $g(6)=112123123412345123456$. How many digits are there in the number $g(64)$? (4 marks)

2

5. 整數 $1^1 \times 2^2 \times 3^3 \times \cdots \times 150^{150}$ 的末尾有多少個零？ （5分）

   How many ending zeros are there in the integer $1^1 \times 2^2 \times 3^3 \times \cdots \times 150^{150}$? (5 marks)

6. 在以下方程組中求 $x+y+z$ 的最大可能值。 （5分）

   Find the maximum possible value of $x+y+z$ in the following system of equations.

   (5 marks)

   $$\begin{cases} x^2 - (y-z)x - yz = 0 \\ y^2 - \left(\dfrac{8}{z^2} - x\right)y - \dfrac{8x}{z^2} = 0 \\ z^2 - (x-y)z - xy = 0 \end{cases}$$

7. 若要從 1、2、⋯、100 中選出兩個不同的數，使得其中一個可被 4 整除，另一個可被 6 整除，問共有多少種不同的選法？ （5分）

   In how many different ways can we choose two different numbers from 1, 2, …, 100 so that one of the chosen numbers is divisible by 4 and the other is divisible by 6? (5 marks)

8. 設 $A = 3^{2008} - 3^{2007} + 3^{2006} - 3^{2005} + \cdots + 3^2 - 3^1$、$B = 3^{12} - 3^{11} + 3^{10} - 3^9 + \cdots + 3^2 - 3^1$。求 A 和 B 的最大公因數。 （5分）

   Let $A = 3^{2008} - 3^{2007} + 3^{2006} - 3^{2005} + \cdots + 3^2 - 3^1$ and $B = 3^{12} - 3^{11} + 3^{10} - 3^9 + \cdots + 3^2 - 3^1$. Find the H.C.F. of A and B. (5 marks)

9. 一輛計程車、一輛小巴和一輛巴士各自在一條直線道路上以均速行駛。已知計程車、小巴和巴士分別於正午、下午 6 時和下午 9 時駛經道路上的某點 X，且計程車與巴士在下午 2 時相遇、計程車與小巴在下午 3 時相遇、小巴與巴士在正午後的 n 小時相遇。求 n。 （6分）

   On a straight road a taxi, a minibus and a bus were each travelling at constant speeds. The taxi, the minibus and the bus passed a point X on the road at noon, 6 pm and 9 pm respectively. Furthermore, the taxi met the bus at 2 pm, the taxi met the minibus at 3 pm and the minibus met the bus n hours after noon. Find n. (6 marks)

10. 若一個正整數可通過把另一正整數連續寫兩次或以上而得到，則此正整數稱為「重複數」。例如：1212、88888、454545 都是「重複數」，7、1001、12344321 則不是「重複數」。求可被 273 整除的最小「重複數」。 （6分）
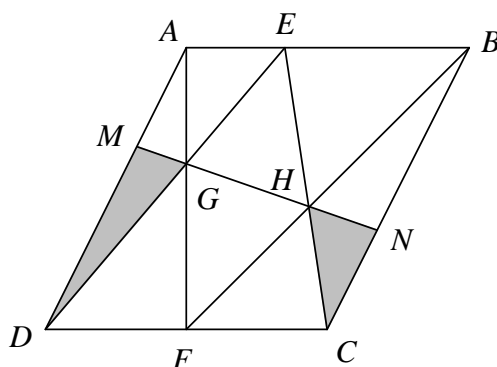
A positive integer is said to be 'repetitive' if it can be formed by repeatedly writing some other positive integer two or more times. For example, 1212, 88888, 454545 are 'repetitive' while 7, 1001, 12344321 are not. Find the smallest 'repetitive' number that is divisible by 273. (6 marks)

11. 若 $\begin{cases} 2\sin^2 A - 2\tan A \sin A + 2 - \sin^2 A - \cos^2 B = 0 \\ 2\cos^2 B - 2\tan A \cos B + 2 - \sin^2 A - \cos^2 B = 0 \end{cases}$ 而 $\sin A \neq \cos B$，求 $\sin^2 A$ 的值。 （6分）

If $\begin{cases} 2\sin^2 A - 2\tan A \sin A + 2 - \sin^2 A - \cos^2 B = 0 \\ 2\cos^2 B - 2\tan A \cos B + 2 - \sin^2 A - \cos^2 B = 0 \end{cases}$ and $\sin A \neq \cos B$, find the value of $\sin^2 A$. (6 marks)

12. 圖中，$ABCD$ 是平行四邊形，E 是 AB 上的一點使得 $AE:EB=1:2$，F 是 CD 的中點，AF 交 DE 於 G，BF 交 CE 於 H。GH 向兩端延長後分別與 AD 和 BC 相交於 M 和 N。若 $ABCD$ 的面積爲 420，求陰影部分的面積之和。 （6分）

In the figure, *ABCD* is a parallelogram. *E* is a point on *AB* such that $AE:EB=1:2$, *F* is the mid-point of *CD*, *AF* meets *DE* at *G* and *BF* meets *CE* at *H*. When produced on both sides, *GH* meets *AD* at *M* and *BC* at *N*. If the area of *ABCD* is 420, find the sum of the shaded areas. (6 marks)

13. 在所示的算式中，不同的字母代表不同的數字。
已知 U = 6，求 CHING 代表的五位數。 （6分）

In the calculation shown, different letters represent different digits. Given that U = 6, find the five-digit number represented by CHING.

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    M  A  T  H
          A  T
 +     P  U  I
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    C  H  I  N  G
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(6 marks)

14. 小娜有兩顆分別是紅色和藍色的骰子。她希望以這兩顆骰子來顯示每天的日期中的「日」。當「日」小於 10 時，則以 0 顯示十位（例如：3 月 8 日時顯示「08」）。假設寫上「6」和「9」的面可以互相對調使用，而且每個面只可寫一個數字，那麼共有多少種方法在兩顆骰子共 12 個面上寫數字？（兩種寫數字的方法中，若可適當地旋轉骰子使得相同顏色的骰子的對應面上的數字相同，則它們視為同一種方法，而「6」和「9」視為相同。） （6 分）

Ivana has a red die and a blue die, and she plans to use them to display the 'day' of the date every day. When the 'day' is smaller than 10, the tens digit will be displayed as '0'. (For example, '08' is displayed on 8th March.) Assuming that faces marked '6' and '9' can be used interchangeably, and that only one digit may be written on each face, in how many different ways can she write the digits on the total of 12 faces on the two dice? (Two ways are considered to be the same if the dice can be rotated so that corresponding faces of the dice of the same colour are marked with the same digits. The digits '6' and '9' are regarded to be the same.) (6 marks)

15. 如果一個四位數的每位數字皆不是 0，而四個數字之積是 9 的倍數，那麼這個四位數稱為「九倍數」。例如，因為 $9 \times 1 \times 3 \times 5 = 135$ 是 9 的倍數，故 9135 是「九倍數」；而 $2 \times 8 \times 3 \times 1 = 48$ 不是 9 的倍數，故 2831 不是「九倍數」。問共有多少個「九倍數」？ （6 分）

A four-digit number is called a 'multi-9 number' if none of its digits is 0 and the product of the 4 digits is a multiple of 9. For example, since $9 \times 1 \times 3 \times 5 = 135$ is a multiple of 9, 9135 is a 'multi-9 number'; since $2 \times 8 \times 3 \times 1 = 48$ is not a multiple of 9, 2831 is not a 'multi-9 number'. How many 'multi-9 numbers' are there? (6 marks)

16. 一個機械人可經遙控器操控在坐標平面上移動。遙控器上共有六個按鈕，分別是「向上平移 1 單位」、「向下平移 1 單位」、「向左平移 1 單位」、「向右平移 1 單位」、「繞原點順時針旋轉 90°」和「繞原點反時針旋轉 90°」。機械人原先在 P 點，按下三次按鈕後它的位置是 (5, 6)。問 P 點的位置有多少個不同的可能？ （7 分）

A robot can be instructed to move on the coordinate plane via a remote control. There are six buttons on the remote control, namely, 'translating 1 unit upward', 'translating 1 unit downward', 'translating 1 unit to the left', 'translating 1 unit to the right', 'rotating clockwise through 90° about the origin' and 'rotating anticlockwise through 90° about the origin'. Initially, the robot is at point P. After pressing buttons three times, the robot is at (5, 6). How many different possibilities are there for the position of P? (7 marks)

正整數可以羅馬數字來表示。羅馬數字由七個字母 I、V、X、L、C、D、M 組成。下表顯示小於 3000 的正整數的羅馬數字表示式，例如：2008 的表示式為 MMVIII、1988 為 MCMLXXXVIII、700 則為 DCC。若某整數的表示式上下倒轉看時仍代表某整數，則原來的整數稱為「好數」。例如：XXX 倒轉看時仍是「XXX」，所以 30 是「好數」；XXI 倒轉看時變成「IXX」，不是某整數的羅馬數字表示式，所以 21 不是「好數」；IV 倒轉看時變成「ΛI」，所以 4 也不是「好數」。

Positive integers can be represented by Roman numerals. Roman numerals are made up of the seven letters I, V, X, L, C, D, M. The table below lists the Roman numeral representations of positive integers less than 3000. For instance, 2008 is written as MMVIII, 1988 is written as MCMLXXXVIII while 700 is written as DCC. A positive integer is said to be 'good' if its Roman numeral representation is still the representation of some integer when read upside down. For instance, 30 is 'good' since XXX still reads 'XXX' when upside down; 21 is not 'good' since XXI reads 'IXX' when upside down, not representing any integer; and 4 is not 'good' since IV reads 'ΛI' when upside down.

	1	2	3	4	5	6	7	8	9
千位 Thousands	M	MM							
百位 Hundreds	C	CC	CCC	CD	D	DC	DCC	DCCC	CM
十位 Tens	X	XX	XXX	XL	L	LX	LXX	LXXX	XC
個位 Unit	I	II	III	IV	V	VI	VII	VIII	IX

17. 某「好數」m 的羅馬數字表示式上下倒轉看時變成了整數 n 的羅馬數字表示式。求 m 和 n 之差的最大可能值。 （3分）

   When read upside down, the Roman numeral representation of a 'good' integer m becomes the Roman numeral representation of the integer n. What is the maximum possible value of the difference between m and n? (3 marks)

18. 在首 2008 個正整數中，有多少個的羅馬數字表示式包含字母「C」？ （5分）

   How many of the first 2008 positive integers have their Roman numeral representation consisting of the letter 'C'? (5 marks)

細閱以下資料，然後回答第 **19** 和第 **20** 題。

Study the following information and answer Questions 19 and 20.

圖中的立體有 14 個面，其中包括 6 個正方形和 8 個正三角形（頂和底都是三角形）。立體每邊的長度都是 1，每個正方形面都和 4 個三角形面相鄰，而每個三角形面都和 3 個正方形面相鄰。

The solid in the figure consists of 14 faces, including 6 squares and 8 equilateral triangles (both the top and bottom faces are triangles). Each edge of the solid has length 1, each square face is adjacent to 4 triangular faces and each triangular face is adjacent to 3 square faces.
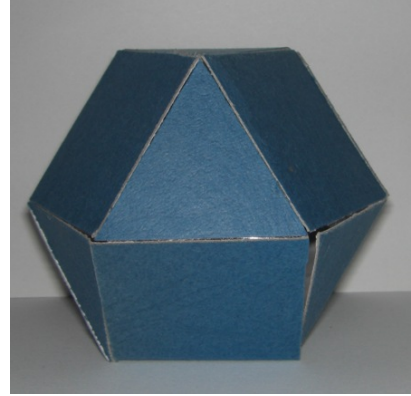
19. 求立體的頂點數目。 （4分）

    Find the number of vertices of the solid. (4 marks)

20. 求立體的體積。 （6分）

    Find the volume of the solid. (6 marks)

全卷完

END OF PAPER