第七屆培正數學邀請賽
7th Pui Ching Invitational Mathematics Competition

決賽（中一組）
Final Event (Secondary 1)

時限：2 小時
Time allowed: 2 hours

參賽者須知：
Instructions to Contestants:

1. 本卷共設 20 題，總分爲 100 分。
   There are 20 questions in this paper and the total score is 100.

2. 除特別指明外，本卷內的所有數均爲十進制。
   Unless otherwise stated, all numbers in this paper are in decimal system.

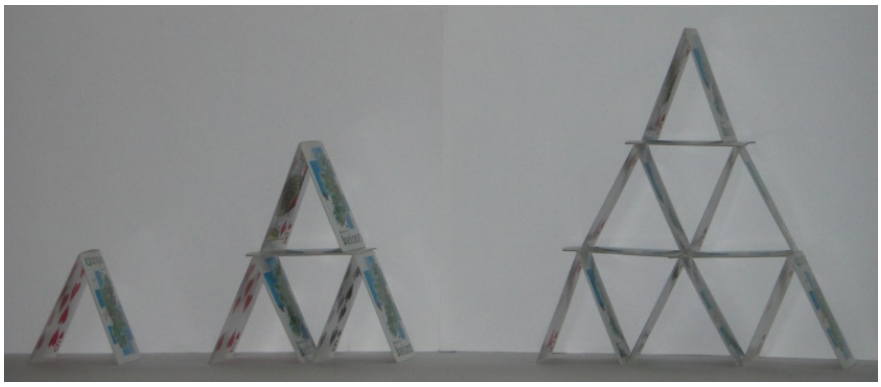
3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。
   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。
   Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。
   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。
   The diagrams in this paper are not necessarily drawn to scale.

1. 下圖顯示以紙牌砌成的「紙牌金字塔」。如圖所示，砌出一層、兩層和三層的金字塔分別需要 2 張、7 張和 15 張紙牌。若依此規律砌出一個 100 層的金字塔，共需要多少張紙牌？ （3分）



The above figure shows some 'card pyramids'. As shown in the figure, a pyramid with 1, 2 and 3 layers requires 2, 7 and 15 cards respectively. If we follow the pattern to build a card pyramid with 100 layers, how many cards are needed? (3 marks)

2. 設 [x] 代表不超過 x 的最大整數，例如 [1.1] = 1、[6.9] = 6 和 [5] = 5。求

$$\left[\frac{2008}{1}\right]-\left[\frac{2007}{1}\right]+\left[\frac{2008}{2}\right]-\left[\frac{2007}{2}\right]+\left[\frac{2008}{3}\right]-\left[\frac{2007}{3}\right]+\cdots+\left[\frac{2008}{2008}\right]-\left[\frac{2007}{2008}\right]$$

的值。 （4分）

Let [x] denote the greatest integer not exceeding x. For example, [1.1] = 1, [6.9] = 6 and [5] = 5. Find the value of

$$\left[\frac{2008}{1}\right]-\left[\frac{2007}{1}\right]+\left[\frac{2008}{2}\right]-\left[\frac{2007}{2}\right]+\left[\frac{2008}{3}\right]-\left[\frac{2007}{3}\right]+\cdots+\left[\frac{2008}{2008}\right]-\left[\frac{2007}{2008}\right].$$ (4 marks)

3. 對於正整數 n，設 $f(n)$ 爲順序列出 1 至 n 所得的正整數，例如：$f(3)=123$、$f(11)=1234567891011$ 等；$g(n)$ 爲順序列出 $f(1)$ 至 $f(n)$ 所得的正整數，例如：$g(3)=112123$、$g(6)=112123123412345123456$ 等。問 $g(64)$ 有多少個數字？ （4分）

For positive integer n, let $f(n)$ be the positive integer formed by listing 1 to n in order, e.g. $f(3)=123$ and $f(11)=1234567891011$; let also $g(n)$ be the positive integer formed by listing $f(1)$ to $f(n)$ in order, e.g. $g(3)=112123$ and $g(6)=112123123412345123456$. How many digits are there in the number $g(64)$? (4 marks)

4. 設 $a_1 = 1$、$a_2 = \frac{5}{2}$，且對任意大於 1 的整數 n 皆有 $a_{n+1} = a_n + \frac{2a_{n-1}}{n-1}$。求 a_{100}。 （4 分）

Let $a_1 = 1$, $a_2 = \frac{5}{2}$ and $a_{n+1} = a_n + \frac{2a_{n-1}}{n-1}$ for all integers n greater than 1. Find a_{100}. (4 marks)

5. 20080308 名參賽者排成一列。現在按以下的方法淘汰參賽者：首先，從隊首將他們順序由 1 開始以整數順序編號，把所有偶數號的參賽者淘汰。然後，把未被淘汰的人按剛才的隊伍順序排列，重新由 1 開始以整數順序編號，把所有編號並非 5 的倍數的參賽者淘汰。如是者，不斷重覆上述的步驟，梅花間竹地淘汰編號是偶數和並非 5 的倍數的參賽者，直至只餘下一名參賽者爲止。最後餘下的參賽者第一次排隊時的編號是甚麼？ （4 分）

20080308 contestants align in a row. Some contestants are eliminated with the following procedures. First number each contestant with integers starting from 1 in ascending order, and all contestants with even numbers are eliminated. The remaining contestants are then aligned in the previous order and re-numbered with integers from 1 onward again. Contestants whose numbers are not multiples of 5 are eliminated. The above procedures are repeated, alternately eliminating contestants with even numbers and with numbers not divisible by 5, until only one contestant remains. What is the number of the remaining contestant during the first round? (4 marks)

6. 某班共有 40 名學生，學號爲 1 至 40。他們測驗時都需要在試卷寫上自己的學號。某天測驗後收回試卷，班長把 40 份試卷上的學號加起來，發現它們之和是 k 的倍數，其中 k 是正整數。當老師得知 k 的值，並知道每位學生寫下的學號都是一個 1 至 40 當中的整數時，他立即斷定最少 2 名學生寫錯了學號。求 k 的最小可能值。 （5 分）

There are 40 students in a class, numbered 1 to 40. They were required to write their class numbers on the test paper during a test. After collecting the test papers after the test was over, the monitor added up the class numbers written on the 40 test papers, and discovered that the sum of the class numbers is a multiple of k, where k is a positive integer. When the teacher was told the value of k and the fact that every class number written was an integer from 1 to 40, he immediately deduced that at least 2 students had written a wrong class number. Find the least possible value of k. (5 marks)

7. 設 $A = 3^{2008} - 3^{2007} + 3^{2006} - 3^{2005} + \cdots + 3^2 - 3^1$ 、 $B = 3^{12} - 3^{11} + 3^{10} - 3^9 + \cdots + 3^2 - 3^1$ 。求 A 和 B 的最大公因數。 （5 分）

Let $A = 3^{2008} - 3^{2007} + 3^{2006} - 3^{2005} + \cdots + 3^2 - 3^1$ and $B = 3^{12} - 3^{11} + 3^{10} - 3^9 + \cdots + 3^2 - 3^1$. Find the H.C.F. of A and B. (5 marks)

8. 某國家通用的硬幣有四種，面值分別爲 2 元、3 元、5 元和 7 元；通用的紙幣有兩種，面值分別爲 500 元和 2008 元。若有每種硬幣各一個與及每種紙幣各兩張，則可以組成多少個不同的正數金額？ （5 分）

In a certain country, there are four types of coins, of denominations \$2, \$3, \$5 and \$7 respectively, and two types of banknotes of denominations \$500 and \$2008 respectively. If one has one coin of each type and two banknotes of each type, how many different positive amounts can be formed? (5 marks)

9. 在所示的算式中，不同的字母代表不同的數字。已知 U = 6，求 CHING 代表的五位數。 （6 分）

In the calculation shown, different letters represent different digits. Given that U = 6, find the five-digit number represented by CHING. (6 marks)

$$\begin{array}{cccccc} & M & A & T & H \\ & & & A & T \\ + & & P & U & I \\ \hline C & H & I & N & G \end{array}$$

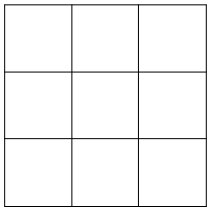
10. 下圖中有一條長 200 m 的軌道，軌道的兩端各有一塊板和一個球。開始時，球 A 以 x m/s 的速度向右走，而球 B 以 5 m/s 的速度向左走。每當兩個球相撞時，它們均即時轉向反方向走，且速度減低三分之二；每當球與板相撞時，球亦即時轉向反方向走，但速度增加 50%。若球 A 走了 2200 m 後其速度爲 1 m/s，求 x。（球的半徑可略去不計。） （6 分）

In the above figure, there is a track of length 200 m. At each of the two ends of the track there is a barrier and a ball. Initially, ball A moves to the right at x m/s while ball B moves to the left at 5 m/s. Whenever the two balls collide, they immediately move in the opposite direction with speeds reduced by two-thirds; whenever a ball hits the barrier, it also turns immediately to move in the opposite direction but with the speed increased by 50%. If the speed of ball A is 1 m/s after it has travelled a total distance of 2200 m, find x. (The radius of the ball is negligible.) (6 marks)

11. 一輛計程車、一輛小巴和一輛巴士各自在一條直線道路上以均速行駛。已知計程車、小巴和巴士分別於正午、下午 6 時和下午 9 時駛經道路上的某點 X，且計程車與巴士在下午 2 時相遇、計程車與小巴在下午 3 時相遇、小巴與巴士在正午後的 n 小時相遇。求 n。（6 分）

On a straight road a taxi, a minibus and a bus were each travelling at constant speeds. The taxi, the minibus and the bus passed a point X on the road at noon, 6 pm and 9 pm respectively. Furthermore, the taxi met the bus at 2 pm, the taxi met the minibus at 3 pm and the minibus met the bus n hours after noon. Find n. (6 marks)

12. 現要在圖中的 3×3 表格的九格中分別填上 1 至 9，使得任何兩個相鄰的方格中的兩數的最大公因數均爲 1。問共有多少種不同的方法填數？（6 分）



The numbers 1 to 9 are to be filled into the nine cells of the 3×3 table in the figure in a way such that numbers in adjacent cells must have an H.C.F. of 1. In how many different ways can we fill in the numbers? (6 marks)

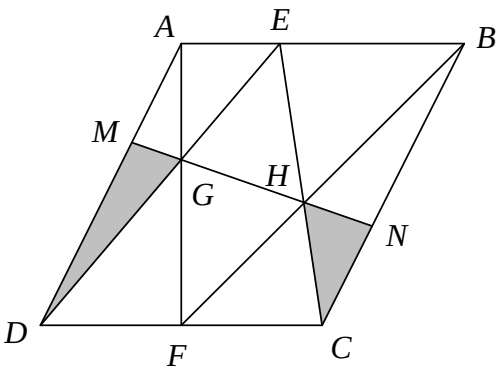
13. 若要從一個正 20 邊形中選出一對互相平行的對角線，共有多少種不同的選法？（6 分）

In how many different ways can a pair of parallel diagonals be chosen in a regular 20-sided polygon? (6 marks)

14. 小娜有兩顆分別是紅色和藍色的骰子。她希望以這兩顆骰子來顯示每天的日期中的「日」。當「日」小於 10 時，則以 0 顯示十位（例如：3 月 8 日時顯示「08」）。假設寫上「6」和「9」的面可以互相對調使用，而且每個面只可寫一個數字，那麼共有多少種方法在兩顆骰子共 12 個面上寫數字？（兩種寫數字的方法中，若可適當地旋轉骰子使得相同顏色的骰子的對應面上的數字相同，則它們視爲同一種方法，而「6」和「9」視爲相同。）（6 分）

Ivana has a red die and a blue die, and she plans to use them to display the ‘day’ of the date every day. When the ‘day’ is smaller than 10, the tens digit will be displayed as ‘0’. (For example, ‘08’ is displayed on 8th March.) Assuming that faces marked ‘6’ and ‘9’ can be used interchangeably, and that only one digit may be written on each face, in how many different ways can she write the digits on the total of 12 faces on the two dice? (Two ways are considered to be the same if the dice can be rotated so that corresponding faces of the dice of the same colour are marked with the same digits. The digits ‘6’ and ‘9’ are regarded to be the same.) (6 marks)

15. 圖中， *ABCD* 是平行四邊形，*E* 是 *AB* 上的一點使得 $AE:EB=1:2$ ，*F* 是 *CD* 的中點，*AF* 交 *DE* 於 *G*，*BF* 交 *CE* 於 *H*。*GH* 向兩端延長後分別與 *AD* 和 *BC* 相交於 *M* 和 *N*。若 *ABCD* 的面積爲 420，求陰影部分的面積之和。



（6 分）

In the figure, *ABCD* is a parallelogram. *E* is a point on *AB* such that $AE:EB=1:2$, *F* is the mid-point of *CD*, *AF* meets *DE* at *G* and *BF* meets *CE* at *H*. When produced on both sides, *GH* meets *AD* at *M* and *BC* at *N*. If the area of *ABCD* is 420, find the sum of the shaded areas.

(6 marks)

16. 一個機械人可經遙控器操控在坐標平面上移動。遙控器上共有六個按鈕，分別是「向上平移 1 單位」、「向下平移 1 單位」、「向左平移 1 單位」、「向右平移 1 單位」、「繞原點順時針旋轉 90°」和「繞原點反時針旋轉 90°」。機械人原先在 *P* 點，按下三次按鈕後它的位置是 (5, 6)。問 *P* 點的位置有多少個不同的可能？ （7 分）

A robot can be instructed to move on the coordinate plane via a remote control. There are six buttons on the remote control, namely, ‘translating 1 unit upward’, ‘translating 1 unit downward’, ‘translating 1 unit to the left’, ‘translating 1 unit to the right’, ‘rotating clockwise through 90° about the origin’ and ‘rotating anticlockwise through 90° about the origin’. Initially, the robot is at point *P*. After pressing buttons three times, the robot is at (5, 6). How many different possibilities are there for the position of *P*?

(7 marks)

細閱以下資料，然後回答第 17 和第 18 題。

Study the following information and answer Questions 17 and 18.

正整數可以羅馬數字來表示。羅馬數字由七個字母 I、V、X、L、C、D、M 組成。下表顯示小於 3000 的正整數的羅馬數字表示式，例如：2008 的表示式爲 MMVIII、1988 爲 MCMLXXXVIII、700 則爲 DCC。若某整數的表示式上下倒轉看時仍代表某整數，則原來的整數稱爲「好數」。例如：XXX 倒轉看時仍是「XXX」，所以 30 是「好數」；XXI 倒轉看時變成「IXX」，不是某整數的羅馬數字表示式，所以 21 不是「好數」；IV 倒轉看時變成「ΛI」，所以 4 也不是「好數」。

Positive integers can be represented by Roman numerals. Roman numerals are made up of the seven letters I, V, X, L, C, D, M. The table below lists the Roman numeral representations of positive integers less than 3000. For instance, 2008 is written as MMVIII, 1988 is written as MCMLXXXVIII while 700 is written as DCC. A positive integer is said to be 'good' if its Roman numeral representation is still the representation of some integer when read upside down. For instance, 30 is 'good' since XXX still reads 'XXX' when upside down; 21 is not 'good' since XXI reads 'IXX' when upside down, not representing any integer; and 4 is not 'good' since IV reads 'ΛI' when upside down.

	1	2	3	4	5	6	7	8	9
千位 Thousands	M	MM							
百位 Hundreds	C	CC	CCC	CD	D	DC	DCC	DCCC	CM
十位 Tens	X	XX	XXX	XL	L	LX	LXX	LXXX	XC
個位 Unit	I	II	III	IV	V	VI	VII	VIII	IX

17. 在首 2008 個正整數中，有多少個是「好數」？ （4 分）

How many of the first 2008 positive integers are 'good'? (4 marks)

18. 在首 2008 個正整數中，有多少個的羅馬數字表示式包含全部 7 個字母？ （5 分）

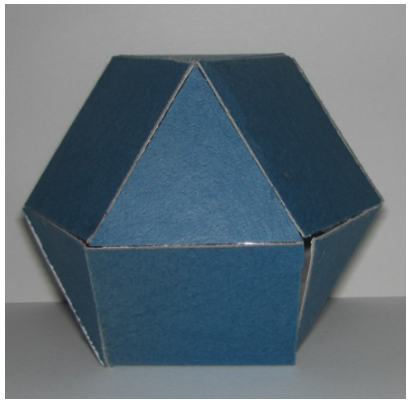
How many of the first 2008 positive integers have their Roman numeral representation consisting of all 7 letters? (5 marks)

細閱以下資料，然後回答第 19 和第 20 題。

Study the following information and answer Questions 19 and 20.

圖中的立體有 14 個面，其中包括 6 個正方形和 8 個正三角形（頂和底都是三角形）。立體每邊的長度都是 1，每個正方形面都和 4 個三角形面相鄰，而每個三角形面都和 3 個正方形面相鄰。

The solid in the figure consists of 14 faces, including 6 squares and 8 equilateral triangles (both the top and bottom faces are triangles). Each edge of the solid has length 1, each square face is adjacent to 4 triangular faces and each triangular face is adjacent to 3 square faces.



19. 求立體的頂點數目。 （4 分）

Find the number of vertices of the solid. (4 marks)

20. 若要從立體中選取兩個相鄰的面，共有多少種選法？ （4 分）

In how many ways can we choose two adjacent faces from the solid? (4 marks)

全卷完

END OF PAPER