第六屆培正數學邀請賽

6th Pui Ching Invitational Mathematics Competition

決賽（高中組）

Final Event (Senior Secondary)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

1. 本卷共設甲、乙兩部分，總分為 100 分。
   This paper is divided into Section A and Section B. The total score is 100.

2. 除特別指明外，本卷內的所有數均為十進制。
   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。
   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。
   Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。
   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。
   The diagrams in this paper are not necessarily drawn to scale.

甲部（75 分）

Section A (75 marks)

1. 某學校有小學部、初中部和高中部。小學部的上課時間表採用 5 天循環週制，分別稱爲上課日 A 至上課日 E（即首五天分別是上課日 A 至上課日 E，第六天又回到上課日 A，如此類推）。初中部的上課時間表採用 7 天循環週制，分別稱爲上課日 F 至上課日 L。高中部的上課時間表則採用 14 天循環週制，分別稱爲上課日 M 至上課日 Z。在學校大門，有一塊牌子寫上當天各部的上課日，例如第一天是「AFM」、第二天是「BGN」，如此類推。若各部的上課日子都相同，那麼牌子上的三個英文字母有多少個不同的組合？（3 分）

   A school comprises a primary school section, a middle school section and a high school section. In the primary school section, the time table runs on a 5-day cycle basis where the schooldays are called Day A to Day E (i.e. the first 5 days are Day A to Day E, the 6th day is back to Day A again and so on). In the middle school section, the time table runs on a 7-day cycle basis where the schooldays are called Day F to Day L. In the high school section, the time table runs on a 14-day cycle basis where the schooldays are called Day M to Day Z. At the entrance of the school there is a board showing the day of school for each section, e.g. 'AFM' for the first day, 'BGN' for the second day and so on. Given that all sections are having school on exactly the same dates, how many different combinations are there for the three letters appearing on the board? (3 marks)

2. 1800 的正因數中，有多少個可被 24 整除？（3 分）

   How many of the positive factors of 1800 are divisible by 24? (3 marks)

3. 有多少個四位正整數的其中一個數字是 0，而其餘三個數字相同？（3 分）

   How many four-digit positive integers have one digit 0 and the other three digits equal? (3 marks)

4. 香港通用的硬幣有七種，面值分別爲 10 元、5 元、2 元、1 元、5 角、2 角和 1 角。若有每種硬幣各一個，則可以組成多少個不同的正數金額？（4 分）

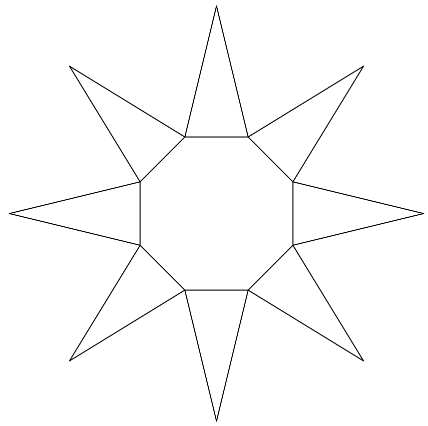
   In Hong Kong, there are seven types of coins, of denominations \$10, \$5, \$2, \$1, 50 cents, 20 cents and 10 cents respectively. If one has one coin of each type, how many different positive amounts can be formed? (4 marks)

5. 若 $x^{x\log x}=10$ 而 $x>1$，求 $\sqrt{x}^{\sqrt{x}}$ 的值。 （4分）

If $x^{x\log x}=10$ where $x>1$, find the value of $\sqrt{x}^{\sqrt{x}}$. (4 marks)

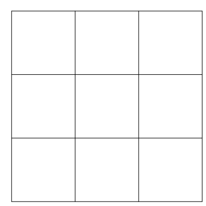
6. 圖中的摺紙圖樣由一個邊長爲 1 的正八邊形和八個底爲 1、高爲 h 的等腰三角形組成。若圖樣不能摺成一個八角錐體，求 h 的最大可能值。 （4分）

The net in the figure is composed of a regular octagon with side length 1 and eight isosceles triangles with base 1 and height h. If the net cannot be folded into an octagonal pyramid, find the maximum possible value of h. (4 marks)



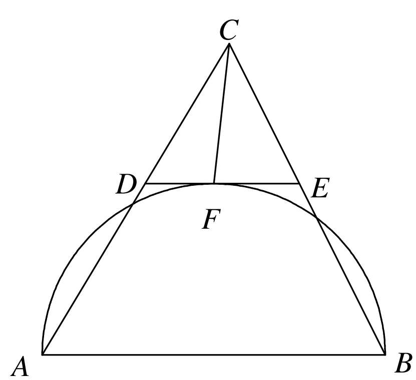
7. 現有一個 3×3 的表格，並要在每格填上 A、B、C 當中的一個字母和 1、2、3 當中的一個數字。若不容許有兩格的字母和數字皆相同，而且每個字母和每個數字都只能在每橫行和每直行出現一次，問共有多少種不同的填法？ （5分）

There is a 3×3 table. In each cell one of the letters A, B, C and one of the digits 1, 2, 3 are to be filled in. If no two cells may have the same letter and digit, and in each row and column each letter and each digit may appear only once, in how many different ways can we fill in the table? (5 marks)



8. 圖中，AB 是半圓的直徑，DE 平行於 AB 且與半圓相切，F 爲切點。AD 與 BE 延長後相交於 C。若 $AB=18$、$DF=3$、$EF=6$，求 CF 的長度。 （5分）

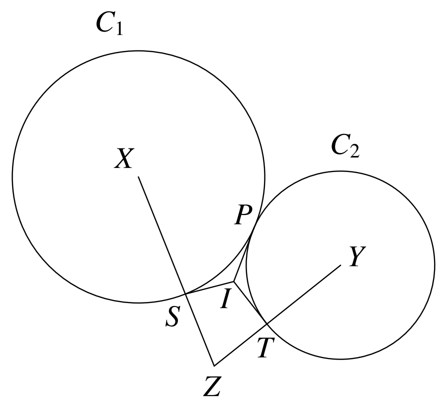
In the figure, AB is a diameter of the semi-circle. DE is parallel to AB and is tangent to the semi-circle at F. AD and BE are produced to meet at C. If $AB=18$, $DF=3$ and $EF=6$, find the length of CF. (5 marks)



9. 求方程 $(90-x)^2+\sin^3 x°=2007$ 所有實數解之和。（5分）

Find the sum of all real solutions to the equation $(90-x)^2+\sin^3 x°=2007$. (5 marks)

10. 圖中，X 和 Y 分別是圓 C_1 和 C_2 的圓心。PI 是兩圓的公切線，P 是切點。IS 切圓 C_1 於 S，IT 切圓 C_2 於 T。XS 和 YT 延長後交於 Z。若 $PI=1$、$XP=3$、$YP=2$，求 ΔXYZ 的面積。（6分）



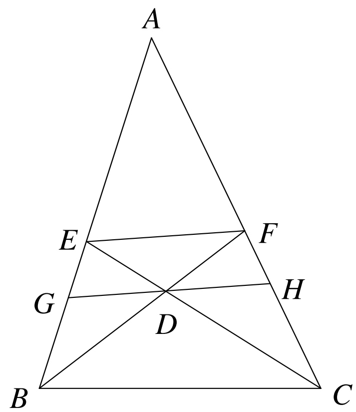
In the figure, X and Y are the centres of circles C_1 and C_2 respectively. PI is a common tangent of the two circles at P, IS is tangent to C_1 at S while IT is tangent to C_2 at T. XS and YT are produced to meet at Z. If $PI=1$, $XP=3$ and $YP=2$, find the area of ΔXYZ. (6 marks)

11. 八位小朋友玩遊戲。他們每人先抽出一張分別寫上整數「1」至「8」的牌，其中他們的牌各不相同。然後他們隨機分成四對再比較他們牌上的數字，每對當中數字較大者可淘汰對手出線進入準決賽。四位準決賽選手再隨機分成兩對作相同的比較，產生兩位決賽選手。最後兩位決賽選手再按相同的方式決定冠軍。假設子祺和英敏分別抽得「5」和「7」，求他們在比賽途中相遇的概率。（6分）

Eight children played a game. Every child drew a card on which an integer from 1 to 8 was written, and their cards were pairwise distinct. They were randomly grouped into 4 pairs and each pair compared the numbers on their cards. The child who had a bigger number in his pair eliminated his opponent and entered the semi-final. The four semi-finalists were randomly grouped into 2 pairs again. The same comparison was done to decide the two finalists. Finally, the champion was decided among the two finalists in the same way. Assuming that Keith and Larry drew the cards ‘5’ and ‘7’ respectively, find the probability that they met during the game. (6 marks)

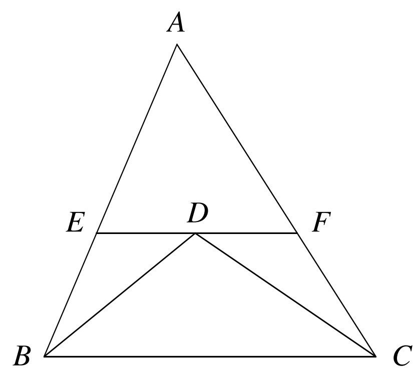
12. 圖中，FB、EC、DG 和 DH 分別是 $\angle ABC$、$\angle ACB$、$\angle EDB$ 和 $\angle FDC$ 的角平分線。EDC、FDB、EGB 和 FHC 都是直線。若 $BE=3$、$CF=2$、$BC=4$，求 $\frac{EG\times FH}{GB\times HC}$。 （6分）

In the figure, FB, EC, DG and DH are the bisectors of $\angle ABC$, $\angle ACB$, $\angle EDB$ and $\angle FDC$ respectively. EDC, FDB, EGB and FHC are straight lines. If $BE=3$, $CF=2$ and $BC=4$, find $\frac{EG\times FH}{GB\times HC}$. (6 marks)



13. 圖中，ABC 是銳角三角形。$\angle ABC$ 和 $\angle ACB$ 的內角平分線交於 D。過 D 作一條平行於 BC 的直線，分別與 AB 和 AC 相交於 E 和 F。若 $AB=15$、$AC=24$，而且 ΔABC 的外接圓面積是 147π，求 EF 的長度。 （7分）

In the figure, ΔABC is acute-angled. The internal bisectors of $\angle ABC$ and $\angle ACB$ meet at D. Through D a straight line parallel to BC is constructed, which meets AB and AC at E and F respectively. If $AB=15$, $AC=24$ and the circumcircle of ΔABC has area 147π, find the length of EF. (7 marks)



14. 某城市的電話號碼全是五位正整數，而且當中任何兩個相鄰的數字最多相差 1。問該市最多有幾個不同的電話號碼？ （7分）

In a city, all telephone numbers are five-digit positive integers in which any two adjacent digits differ by at most 1. At most how many different telephone numbers are there in the city? (7 marks)

15. 已知 n 和 $\frac{n^3-29\times 30\times 31\times 32}{n+1}$ 皆是整數。求 n 所有可能值之和。 （7分）

Given that both n and $\frac{n^3-29\times 30\times 31\times 32}{n+1}$ are integers, find the sum of all possible values of n. (7 marks)

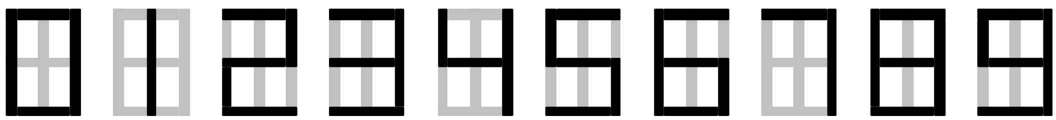
乙部（**25** 分）

Section B (25 marks)

細閱以下資料，然後回答第 16 至第 20 題。

Study the following information and answer Questions 16 to 20.

一家科技公司舉辦了一個爲期五天的計算機展覽，展出該公司的各款計算機。所有計算機的屏幕上都只會顯示小數點和 0 至 9 的數字，而且都使用同一款字體：



爲吸引更多人入場，大會舉辦了一個「我最喜愛的計算機」選舉。每位入場人士均會獲發一張選票，在展出的計算機中選擇一個他們最喜愛的型號。各人交回選票時可獲贈一張「刮刮咭」和一張 50 元優惠券供在場內購物之用。每張「刮刮咭」上有五格，參加者需刮去其中三格，如果這三格的圖案相同便會中獎。而每張優惠券上的條款如下：

$50 優惠券

使用細則：

1. 憑券購物可作 50 元使用。
2. 每次購物最多只可使用優惠券兩張。
3. 以優惠券購物時不能享用特價優惠。
4. 如購物總額低於優惠券面額，餘額不獲發還。

所有入場人士均需在展覽開始前預先登記，並於入場時佩戴該公司的一款「計算機襟章」，同時在襟章的屏幕上打出他們的登記編號以茲識別。每個登記編號都是一個四位正整數，而且各人的登記編號互不相同。

在展覽的第一天，大會發現有些入場人士佩戴計算機襟章時上下倒轉了，因而使襟章上顯示出另一個有效的登記編號，例如：「6681」變成了「1899」。大會於是把這些上下倒轉後屏幕顯示成另一個有效登記編號的四位正整數稱爲「壞數」（例如：6681 和 1899 都是「壞數」，1234 則不是「壞數」），並在第二天起重新發出登記編號，使得所有登記編號都不是「壞數」，而新的登記編號依然符合原先的條件。另一方面，在屏幕上下倒轉後仍顯示原數的四位正整數稱爲「好數」，8888 就是其中一個例子。

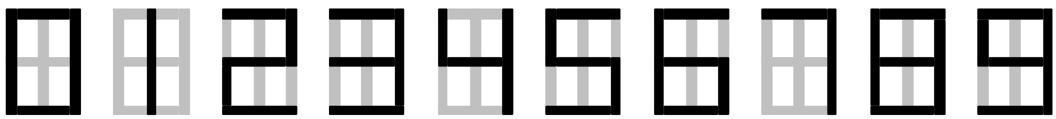
陳先生在展覽的第一天帶同兩名兒子志豪和志強出席，三人就他們的登記編號有一些有趣的發現。以下是他們之間的對話：

陳先生說：「我的登記編號是個『好數』呢。」

志豪說：「我的登記編號是個『壞數』呢。」

志強說：「我的登記編號比爸爸的大 300，比志豪的小 200，既非『好數』亦非『壞數』。」

A technology company had organised a 5-day calculator exhibition displaying the various models of calculators of the company. The screens of all calculators display only the decimal point and digits from 0 to 9 with the same font:



To attract more people, the organiser had included a poll entitled ‘My Favourite Calculator’. Each participant was given a ballot paper so that they could select their favourite model among all calculators displayed. When returning the ballot paper, each participant would be given a ‘scratch card’ and a $50 coupon for purchase in the exhibition. Each ‘scratch card’ consists of five cells; the player scratches three of them and wins a prize if the pictures in the three cells are the same. The terms and conditions of each $50 coupon are as follows:

$50 COUPON

Terms and Conditions:

1. This coupon may be used as $50 for purchase.
2. At most two coupons may be used for each purchase.
3. No discount will be offered when making purchases with this coupon.
4. No return will be given for purchases less than the face value of the coupon.

All participants were required to register before the start of the exhibition, and wear a ‘calculator badge’ produced by the company. They must also show their registration number on the screen of the calculator badge for identification purpose. Each registration number is a 4-digit positive integer, and the participants got pairwise different registration numbers.

During the first day of the exhibition, the organiser found that some participants wore the calculator badge upside down, and as a result a different but legal registration number was shown, e.g. ‘6681’ became ‘1899’. The organiser therefore called those 4-digit positive integers which became another legal registration number on screen when read upside down ‘bad’ numbers. (For instance, 6681 and 1899 are both ‘bad’ while 1234 is not.) On the second day, the organiser issued a different set of registration numbers so that all of them were not ‘bad’ and still satisfied the original conditions. On the other hand, 4-digit integers which read the same when the screen is put upside down are said to be ‘good’, and 8888 is one such example.

On the first day of the exhibition, Mr Chan participated with his two sons, Henry and Ken. They had some interesting discoveries about their registration numbers, and their conversation was as follows:

“My registration number is a ‘good’ number,” said Mr Chan.

“My registration number is a ‘bad’ number,” said Henry.

“My registration number is 300 greater than Dad’s and 200 smaller than Henry’s, and is neither ‘good’ nor ‘bad’,” said Ken.

16. 已知「刮刮咭」上每格的圖案都是電腦隨機印上的，有 50% 機會印上「☆」、30% 機會印上「ϟ」、20% 機會印上「∮」。陳先生隨意刮去他的「刮刮咭」的其中三格，那麼他中獎的機會是多少？ （4分）

Given that the picture in each cell of a ‘scratch card’ is randomly printed by computer, with 50% chance being ‘☆’, 30% chance being ‘ϟ’ and 20% chance being ‘∮’. If Mr Chan randomly scratched three cells on his ‘scratch card’, what is the probability that he could win a prize? (4 marks)

17. 「好數」共有多少個？ （5分）

How many ‘good’ numbers are there? (5 marks)

18. 已知在陳先生和兩名兒子就登記編號的對話中，其中一人說了謊，其餘兩人說了真話，而且三個登記編號中最少有一個的千位不是 9，則三個登記編號之和的最大可能值是甚麼？ （6分）

It is known that in the conversation that Mr Chan had with his two sons regarding the registration numbers, one of them had lied while the other two had told the truth. Furthermore, at least one of the three registration numbers had a thousands digit different from 9. What is the greatest possible sum of the three registration numbers? (6 marks)

19. 場內展出的其中一款計算機有四個特別鍵：「+9」（可把屏幕上的數加 9）、「−1」（可把屏幕上的數減 1）、「log」（若屏幕上的數爲 x，則按鍵後會變成 $\log x$）和「10^x」（若屏幕上的數爲 x，則按鍵後會變成10^x）。現時屏幕上顯示 102400。若依次按下「log」、「+9」、「+9」、「log」、「−1」、「10^x」、「10^x」這七個特別鍵，則屏幕上顯示的數是甚麼？ （5分）

A model of calculator shown in the exhibition contains 4 special keys: ‘+9’ (which adds 9 to the number on screen), ‘−1’ (which subtracts 1 from the number on screen), ‘log’ (which changes the number x on screen to $\log x$) and ‘10^x’ (which changes the number x on screen to 10^x). Now the screen shows 102400. If the 7 special keys ‘log’, ‘+9’, ‘+9’, ‘log’, ‘−1’, ‘10^x’ and ‘10^x’ are pressed in order, what will be the number shown on screen? (5 marks)

20. 場內展出的其中一款計算機只能處理整數。當運算結果不是整數時，計算機會把小數部分截去。例如：計算 $25\div6\times9$ 時，因爲 $25\div6$ 會變成 4，因此最後答案會是 36。志強使用這款計算機來計算 $1000\div n\times n$，其中 n 是小於 1000 的正整數。那麼志強所得的答案有多少個不同的可能值？ （5分）

A model of calculator shown in the exhibition can only handle integers. If the result of a computation is not an integer, the decimal part will be truncated. For instance, $25\div6\times9$ will be computed to be 36 because $25\div6$ becomes 4. Ken used this model of calculator to compute $1000\div n\times n$, where n is a positive integer less than 1000. How many different possible values are there for the result he obtained? (5 marks)

全卷完

END OF PAPER